Lecture 10

Basic Dividers

Required Reading

Behrooz Parhami,
Computer Arithmetic: Algorithms and Hardware Design

Chapter 13, Basic Division Schemes
13.1, Shift/Subtract Division Algorithms
13.3, Restoring Hardware Dividers
13.4, Non-Restoring and Signed Division

Chapter 15 Variation in Dividers
15.6, Combined Multiply/Divide Units

Notation
and
Basic Equations

Notation

z	Dividend	$z_{2k-1} z_{2k-2} \ldots z_2\ z_1\ z_0$
d	Divisor	$d_{k-1} d_{k-2} \ldots d_1\ d_0$
q	Quotient	$q_{k-1} q_{k-2} \ldots q_1\ q_0$
s	Remainder ($s = z - dq$)	$s_{k-1} s_{k-2} \ldots s_1\ s_0$

Basic Equations of Division

$$z = d\,q + s$$

$$|\,s\,| < |\,d\,|$$

$$\text{sign}(s) = \text{sign}(z)$$

$z > 0$

$0 \leq s < |\,d\,|$

$z < 0$

$-|\,d\,| < s \leq 0$

Unsigned Integer Division Overflow

- Must check overflow because obviously the quotient q can also be 2k bits.
  - For example, if the divisor d is 1, then the quotient q is the dividend z, which is 2k bits

Condition for no overflow (i.e. q fits in k bits):

$$z = q\ d + s < (2^k - 1)\ d + d = d\ 2^k$$

$$z = z_H\ 2^k + z_L < d\ 2^k$$

$$z_H < d$$

Sequential Integer Division
Basic Equations

$$s^{(0)} = z$$

$$s^{(j)} = 2\ s^{(j-1)} - q_{k-j}\ (2^k\ d) \quad \text{for } j=1..k$$

$$s^{(k)} = 2^k\ s$$

Sequential Integer Division Justification

$s^{(1)} = 2\ z - q_{k-1}\ (2^k\ d)$

$s^{(2)} = 2(2\ z - q_{k-1}\ (2^k\ d)) - q_{k-2}\ (2^k\ d)$

$s^{(3)} = 2(2(2\ z - q_{k-1}\ (2^k\ d)) - q_{k-2}\ (2^k\ d)) - q_{k-3}\ (2^k\ d)$

. . . . . .

$s^{(k)} = 2(.\ .\ .\ 2(2(2\ z - q_{k-1}\ (2^k\ d)) - q_{k-2}\ (2^k\ d)) - q_{k-3}\ (2^k\ d)\ .\ .\ .$

$\quad - q_0\ (2^k\ d) =$

$\quad = 2^k\ z - (2^k\ d)\ (q_{k-1}\ 2^{k-1} + q_{k-2}\ 2^{k-2} + q_{k-3}\ 2^{k-3} + \ldots + q_0 2^0) =$

$\quad = 2^k\ z - (2^k\ d)\ q = 2^k\ (z - d\ q) = 2^k\ s$

Fig. 13.2 Examples of sequential division with integer and fractional operands.

Integer division		Fractional division	
z	0 1 1 1 0 1 0 1	z_{frac}	. 0 1 1 1 0 1 0 1
$2^4 d$	1 0 1 0	d_{frac}	. 1 0 1 0
$s^{(0)}$	0 1 1 1 0 1 0 1	$s^{(0)}$	. 0 1 1 1 0 1 0 1
$2s^{(0)}$	0 1 1 1 0 1 0 1	$2s^{(0)}$	0 . 1 1 1 0 1 0 1
$-q_3 2^4 d$	1 0 1 0 $\{q_3 = 1\}$	$-q_{-1}d$	. 1 0 1 0 $\{q_{-1}=1\}$
$s^{(1)}$	0 1 0 0 1 0 1	$s^{(1)}$	. 0 1 0 0 1 0 1
$2s^{(1)}$	0 1 0 0 1 0 1	$2s^{(1)}$	0 . 1 0 0 1 0 1
$-q_2 2^4 d$	0 0 0 0 $\{q_2 = 0\}$	$-q_{-2}d$	. 0 0 0 0 $\{q_{-2}=0\}$
$s^{(2)}$	1 0 0 1 0 1	$s^{(2)}$	. 1 0 0 1 0 1
$2s^{(2)}$	1 0 0 1 0 1	$2s^{(2)}$	1 . 0 0 1 0 1
$-q_1 2^4 d$	1 0 1 0 $\{q_1 = 1\}$	$-q_{-3}d$	. 1 0 1 0 $\{q_{-3}=1\}$
$s^{(3)}$	1 0 0 0 1	$s^{(3)}$	. 1 0 0 0 1
$2s^{(3)}$	1 0 0 0 1	$2s^{(3)}$	1 . 0 0 0 1
$-q_0 2^4 d$	1 0 1 0 $\{q_0 = 1\}$	$-q_{-1}d$	. 1 0 1 0 $\{q_{-4}=1\}$
$s^{(4)}$	0 1 1 1	$s^{(4)}$	. 0 1 1 1
s	0 1 1 1	s_{frac}	0 . 0 0 0 0 0 1 1 1
q	1 0 1 1	q_{frac}	. 1 0 1 1

Fractional Division

Unsigned Fractional Division

z_{frac}	Dividend	$.z_{-1}z_{-2} \ldots z_{-(2k-1)}z_{-2k}$
d_{frac}	Divisor	$.d_{-1}d_{-2} \ldots d_{-(k-1)}\, d_{-k}$
q_{frac}	Quotient	$.q_{-1}q_{-2} \ldots q_{-(k-1)}\, q_{-k}$
s_{frac}	Remainder	$\underbrace{.000\ldots0}_{\text{k bits}}s_{-(k+1)} \ldots s_{-(2k-1)}\, s_{-2k}$

Integer vs. Fractional Division

For Integers:

$$z = q\,d + s \quad \Big| \cdot 2^{-2k}$$

$$z\,2^{-2k} = (q\,2^{-k})\,(d\,2^{-k}) + s\,(2^{-2k})$$

For Fractions:

$$z_{frac} = q_{frac}\,d_{frac} + s_{frac}$$

where

$z_{frac} = z\,2^{-2k}$ $\qquad q_{frac} = q\,2^{-k}$

$d_{frac} = d\,2^{-k}$ $\qquad s_{frac} = s\,2^{-2k}$

Unsigned Fractional Division Overflow

Condition for no overflow:

$$z_{frac} < d_{frac}$$

Sequential Fractional Division Basic Equations

$$s^{(0)} = z_{frac}$$

$$s^{(j)} = 2\ s^{(j-1)} - q_{-j}\ d_{frac} \qquad \text{for } j=1..k$$

$$2^k \cdot s_{frac} = s^{(k)}$$

$$s_{frac} = 2^{-k} \cdot s^{(k)}$$

Sequential Fractional Division Justification

$s^{(1)} = 2\ z_{frac} - q_{-1}\ d_{frac}$

$s^{(2)} = 2(2\ z_{frac} - q_{-1}\ d_{frac}) - q_{-2}\ d_{frac}$

$s^{(3)} = 2(2(2\ z_{frac} - q_{-1}\ d_{frac}) - q_{-2}\ d_{frac}) - q_{-3}\ d_{frac}$

. . . . . .

$s^{(k)} = 2(.\ .\ .\ 2(2(2\ z_{frac} - q_{-1}\ d_{frac}) - q_{-2}\ d_{frac}) - q_{-3}\ d_{frac}\ .\ .\ .$
$- q_{-k}\ d_{frac} =$
$= 2^k\ z_{frac} - d_{frac}\ (q_{-1}\ 2^{k-1} + q_{-2}\ 2^{k-2} + q_{-3}\ 2^{k-3} + \ldots + q_{-k}2^0) =$
$= 2^k\ z_{frac} - d_{frac}\ 2^k (q_{-1}\ 2^{-1} + q_{-2}\ 2^{-2} + q_{-3}\ 2^{-3} + \ldots + q_{-k}2^{-k}) =$
$= 2^k\ z_{frac} - (2^k\ d_{frac})\ q_{frac} = 2^k\ (z_{frac} - d_{frac}\ q_{frac}) = 2^k\ s_{frac}$

Restoring Unsigned Integer Division

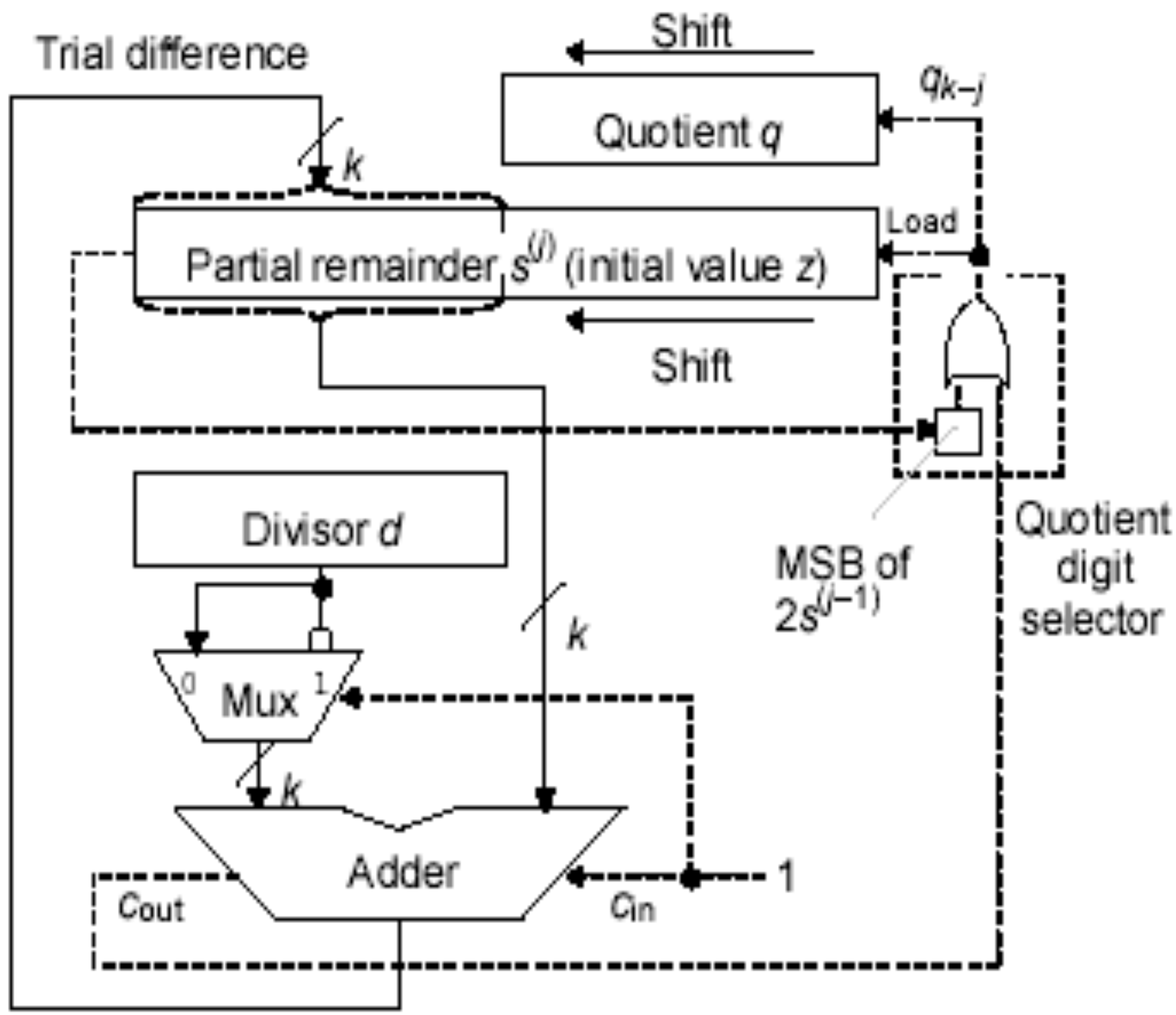
Restoring Unsigned Integer Division

$s^{(0)} = z$
for $j = 1$ to k
  if $2\, s^{(j-1)} - 2^k\, d > 0$
    $q_{k-j} = 1$
    $s^{(j)} = 2\, s^{(j-1)} - 2^k\, d$
  else
    $q_{k-j} = 0$
    $s^{(j)} = 2\, s^{(j-1)}$
end for

Fig. 13.6 Example of restoring unsigned division.

z	0 1 1 1	0 1 0 1	No overflow, since:
$2^4 d$	0 1 0 1 0		$(0111)_{two} < (1010)_{two}$
$-2^4 d$	1 0 1 1 0		
$s^{(0)}$	0 0 1 1 1	0 1 0 1	
$2s^{(0)}$	0 1 1 1 0	1 0 1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(1)}$	0 0 1 0 0	1 0 1	Positive, so set $q_3 = 1$
$2s^{(1)}$	0 1 0 0 1	0 1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(2)}$	1 1 1 1 1	0 1	Negative, so set $q_2 = 0$
$s^{(2)}=2s^{(1)}$	0 1 0 0 1	0 1	and restore
$2s^{(2)}$	1 0 0 1 0	1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(3)}$	0 1 0 0 0	1	Positive, so set $q_1 = 1$
$2s^{(3)}$	1 0 0 0 1		
$+(-2^4 d)$	1 0 1 1 0		
$s^{(4)}$	0 0 1 1 1		Positive, so set $q_0 = 1$
s		0 1 1 1	
q		1 0 1 1	

Fig. 13.5 Shift/subtract sequential restoring divider.



Non-Restoring Unsigned Integer Division

Non-Restoring Unsigned Integer Division

$s^{(0)} = z$

$s^{(1)} = 2\, s^{(0)} - 2^k d$

for $j = 2$ to k

    if $s^{(j-1)} \geq 0$

        $q_{k-(j-1)} = 1$

        $s^{(j)} = 2\, s^{(j-1)} - 2^k d$

    else

        $q_{k-(j-1)} = 0$

        $s^{(j)} = 2\, s^{(j-1)} + 2^k d$

end for

if $s^{(k)} \geq 0$

    $q_0 = 1$

else

    $q_0 = 0$

    Correction step

Non-Restoring Unsigned Integer Division

Correction step

$$s = 2^{-k} \cdot s^{(k)}$$

$$z = q\, d + s$$

$z, q, d \geq 0$ $\qquad$ $s<0$

$$z = (q-1)\, d + (s+d)$$
$$z = q'\, d + s'$$

Example of nonrestoring unsigned division

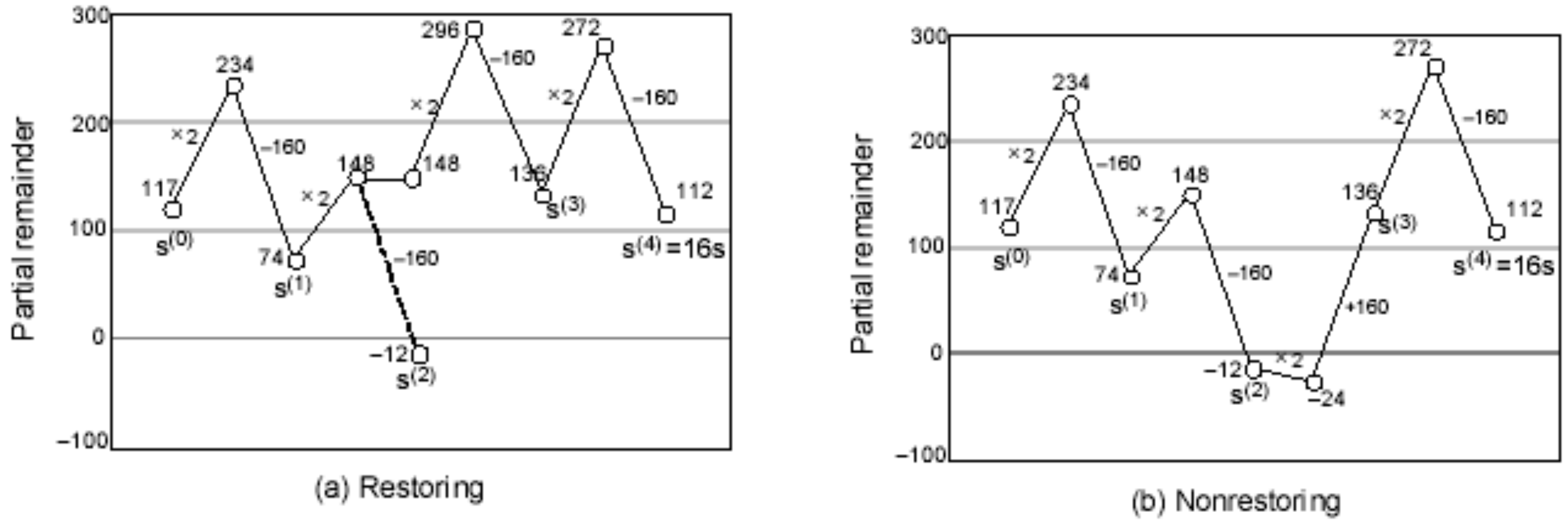
z	0 1 1 1	0 1 0 1	No overflow, since:
2^4d	0 1 0 1 0		$(0111)_{two} < (1010)_{two}$
-2^4d	1 0 1 1 0		
$s^{(0)}$	0 0 1 1 1	0 1 0 1	
$2s^{(0)}$	0 1 1 1 0	1 0 1	Positive,
$+(-2^4d)$	1 0 1 1 0		so subtract
$s^{(1)}$	0 0 1 0 0	1 0 1	
$2s^{(1)}$	0 1 0 0 1	0 1	Positive, so set $q_3 = 1$
$+(-2^4d)$	1 0 1 1 0		and subtract
$s^{(2)}$	1 1 1 1 1	0 1	
$2s^{(2)}$	1 1 1 1 0	1	Negative, so set $q_2 = 0$
$+2^4d$	0 1 0 1 0		and add
$s^{(3)}$	0 1 0 0 0	1	
$2s^{(3)}$	1 0 0 0 1		Positive, so set $q_1 = 1$
$+(-2^4d)$	1 0 1 1 0		and subtract
$s^{(4)}$	0 0 1 1 1		Positive, so set $q_0 = 1$
s		0 1 1 1	
q		1 0 1 1	

Partial remainder variations for restoring and nonrestoring division

Example

$(0\,1\,1\,1\quad 0\,1\,0\,1)_{two}$ / $(1\,0\,1\,0)_{two}$

$(117)_{ten}$ / $(10)_{ten}$



Non-Restoring Unsigned Integer Division

Justification

$s^{(j-1)} \geq 0$

$2\, s^{(j-1)} - 2^k\, d < 0$

$2\, (2\, s^{(j-1)}) - 2^k\, d \geq 0$

Restoring division

$s^{(j)} = 2\, s^{(j-1)}$

$s^{(j+1)} = 2\, s^{(j)} - 2^k\, d =$

$= 4\, s^{(j-1)} - 2^k\, d$

Non-Restoring division

$s^{(j)} = 2\, s^{(j-1)} - 2^k\, d$

$s^{(j+1)} = 2\, s^{(j)} + 2^k\, d =$

$= 2\, (2\, s^{(j-1)} - 2^k\, d) + 2^k\, d =$

$= 4\, s^{(j-1)} - 2^k\, d$

Convergence of the Partial Quotient to q

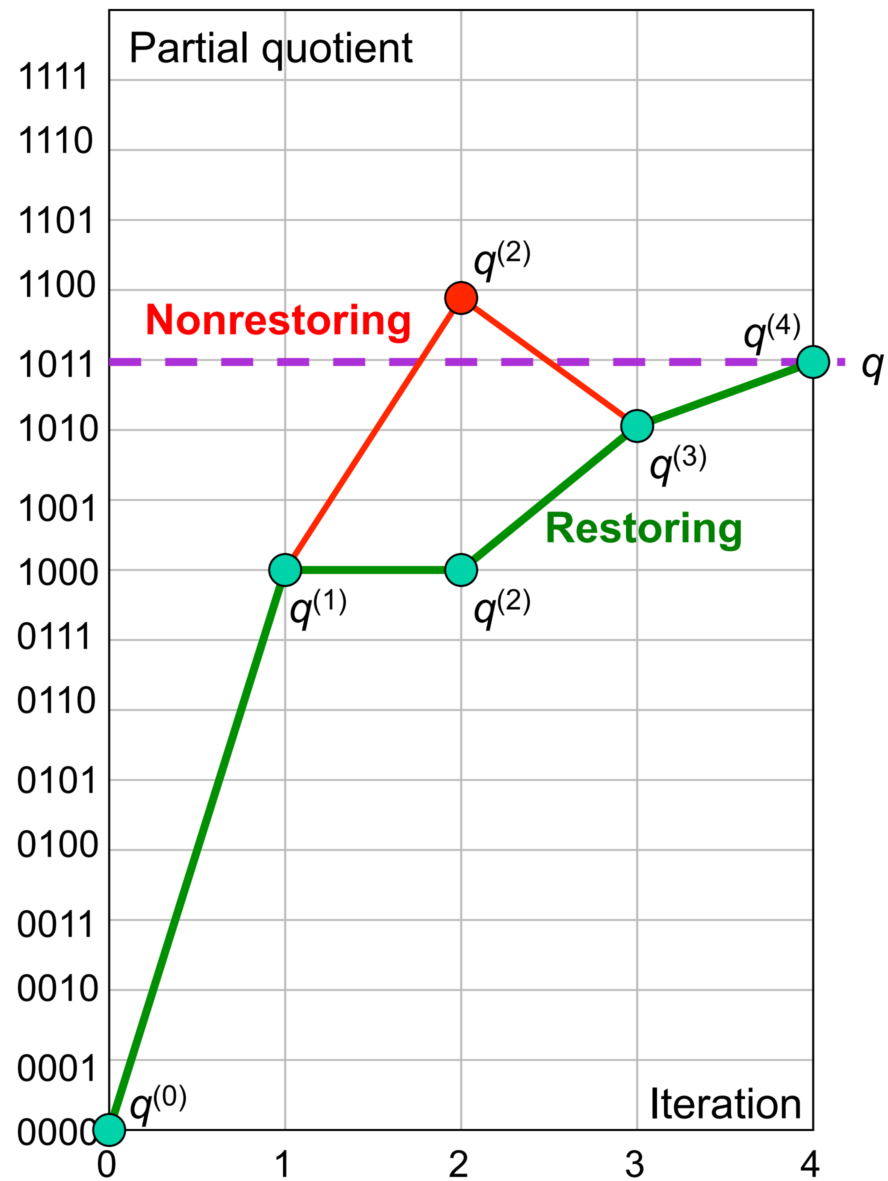
Example

$(0\ 1\ 1\ 1\quad 0\ 1\ 0\ 1)_{two}\ /\ (1\ 0\ 1\ 0)_{two}$

$(117)_{ten}/(10)_{ten} = (11)_{ten} = (1011)_{two}$

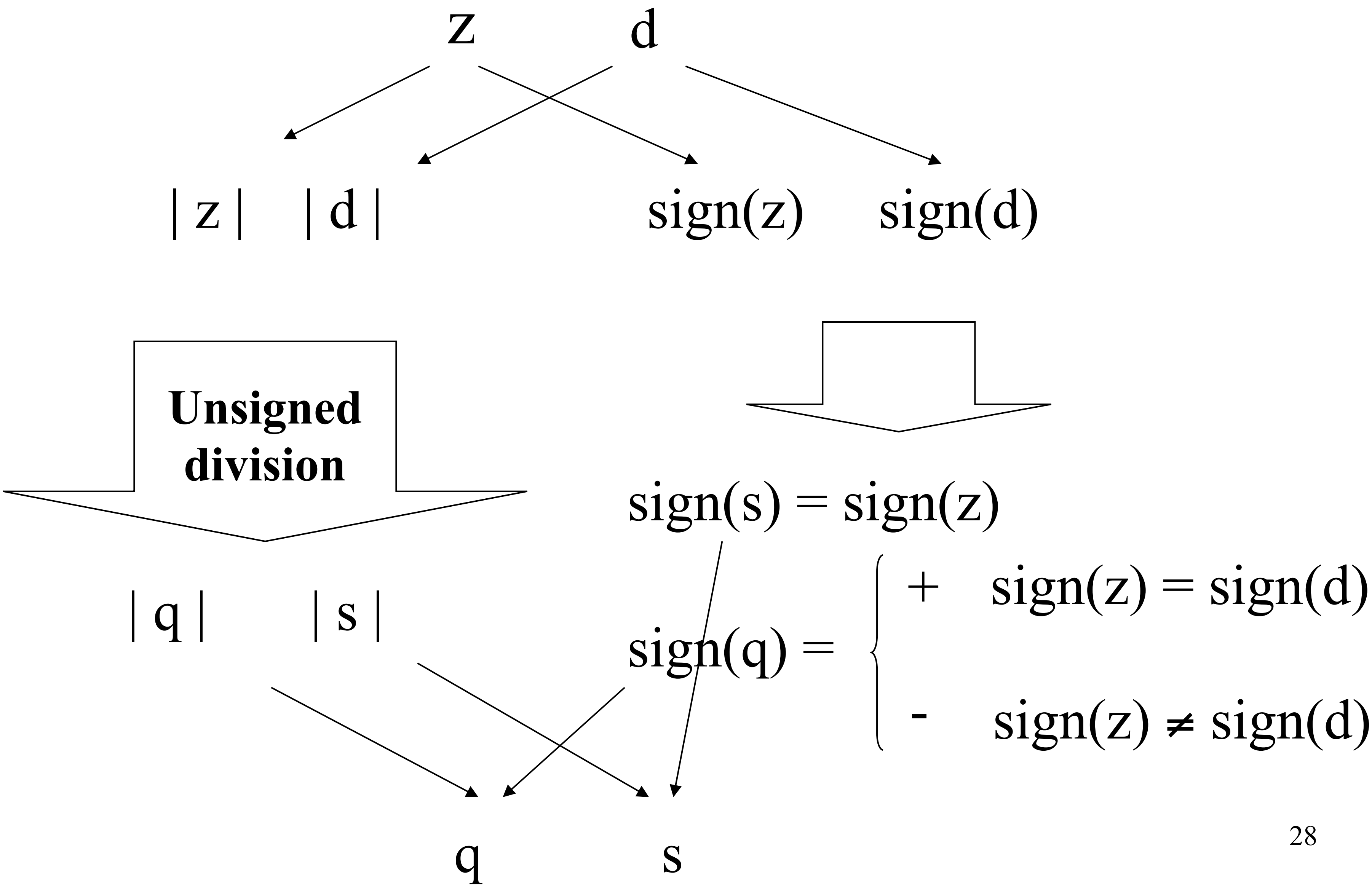
In restoring division, the partial quotient converges to q from below

In nonrestoring division, the partial quotient may overshoot q, but converges to it after some oscillations



Signed Integer Division

Signed Integer Division



Examples of Signed Integer Division

Examples of division with signed operands

$z = 5 \quad d = 3 \quad \Rightarrow \quad q = 1 \quad s = 2$

$z = 5 \quad d = -3 \quad \Rightarrow \quad q = -1 \quad s = 2$

$z = -5 \quad d = 3 \quad \Rightarrow \quad q = -1 \quad s = -2$

$z = -5 \quad d = -3 \quad \Rightarrow \quad q = 1 \quad s = -2$

Magnitudes of q and s are unaffected by input signs
Signs of q and s are derivable from signs of z and d

Non-Restoring
Signed Integer Division

Non-Restoring Signed Integer Division

$s^{(0)} = z$

for $j = 1$ to k

  if $\text{sign}(s^{(j-1)}) == \text{sign}(d)$

    $q_{k-j} = 1$

    $s^{(j)} = 2\, s^{(j-1)} - 2^k d = 2\, s^{(j-1)} - q_{k-j}\,(2^k d)$

  else

    $q_{k-j} = -1$

    $s^{(j)} = 2\, s^{(j-1)} + 2^k d = 2\, s^{(j-1)} - q_{k-j}\,(2^k d)$

end for

q = BSD_2's_comp_conversion(q)

Correction_step

Non-Restoring Signed Integer Division

Correction step

$$s = 2^{-k} \cdot s^{(k)}$$

$$z = q\,d + s$$

$$\text{sign}(s) = \text{sign}(z)$$

$$z = (q-1)\,d + (s+d)$$
$$z = q'\,d + s'$$

$$z = (q+1)\,d + (s-d)$$
$$z = q''\,d + s''$$

Example of nonrestoring signed division

z		0 0 1 0	0 0 0 1	
2^4d	1	1 0 0 1		
-2^4d	0	0 1 1 1		
$s^{(0)}$	0	0 0 1 0	0 0 0 1	
$2s^{(0)}$	0	0 1 0 0	0 0 1	sign($s^{(0)}$) ≠ sign(d), so set $q_3 = {}^{-}1$ and add
$+2^4d$	1	1 0 0 1		
$s^{(1)}$	1	1 1 0 1	0 0 1	
$2s^{(1)}$	1	1 0 1 0	0 1	sign($s^{(1)}$) = sign(d), so set $q_2 = 1$ and subtract
$+(-2^4d)$	0	0 1 1 1		
$s^{(2)}$	0	0 0 0 1	0 1	
$2s^{(2)}$	0	0 0 1 0	1	sign($s^{(2)}$) ≠ sign(d), so set $q_1 = {}^{-}1$ and add
$+2^4d$	1	1 0 0 1		
$s^{(3)}$	1	1 0 1 1	1	
$2s^{(3)}$	1	0 1 1 1		sign($s^{(3)}$) = sign(d), so set $q_0 = 1$ and subtract
$+(-2^4d)$	0	0 1 1 1		
$s^{(4)}$	1	1 1 1 0		sign($s^{(4)}$) ≠ sign(z),
$+(-2^4d)$	0	0 1 1 1		so perform corrective subtraction
$s^{(4)}$	0	0 1 0 1		
s			0 1 0 1	
q			$^{-}1\ 1\ ^{-}1\ 1$	

$p =$		0	1	0	1	Shift, compl MSB
	1	1	0	1	1	Add 1 to correct
		1	1	0	0	Check: $33/(-7) = -4$

BSD → 2's Complement Conversion

$$q = (q_{k-1}\ q_{k-2}\ \ldots\ q_1\ q_0)_{BSD} =$$

$$= (\overline{p_{k-1}}\ p_{k-2}\ \ldots\ p_1\ p_0\ 1)_{2's\ complement}$$

where

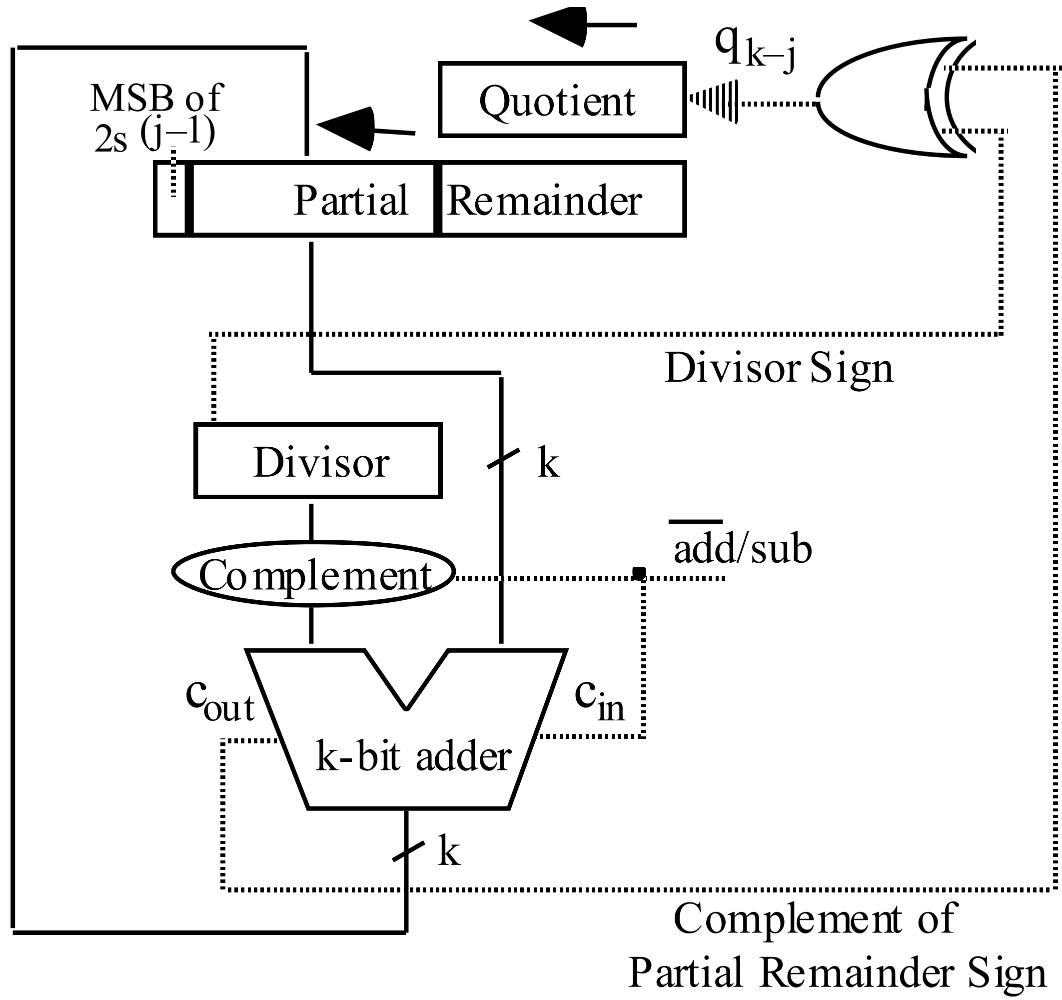
q_i	p_i
-1	0
1	1

Example:

q_{BSD}	1 -1 1 1
p	1 0 1 1
$q_{2'scomp}$	0 0 1 1 1 = 0 1 1 1

no overflow if $p_{k-2} = \overline{p_{k-1}}$ $(q_{k-1} \neq q_{k-2})$

Nonrestoring Hardware Divider



Multiply/Divide Unit

Multiply-Divide Unit

The control unit proceeds through necessary steps for multiplication or division (including using the appropriate shift direction)

The slight speed penalty owing to a more complex control unit is insignificant

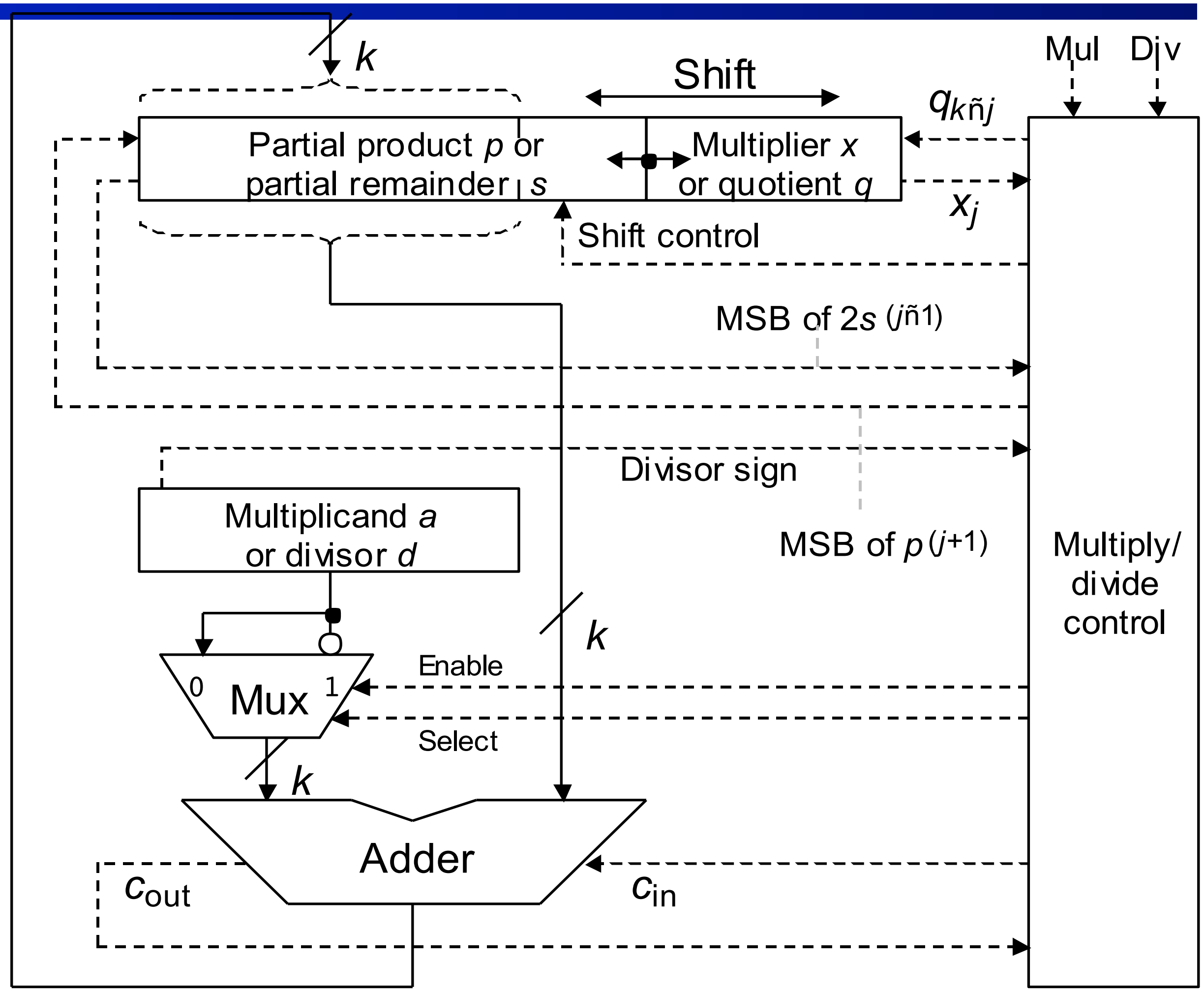


Fig. 15.9 Sequential radix-2 multiply/divide unit.