Lecture 8

FPGA Multipliers

Radix 2 Sequential Multipliers

Required Reading

Behrooz Parhami,
Computer Arithmetic: Algorithms and Hardware Design

Chapter 9, Basic Multiplication Scheme
Chapter 10, High-Radix Multipliers
Chapter 12.3, Bit-Serial Multipliers
Chapter 12.4, Modular Multipliers

FPGA Multipliers

Notation

Y	Multiplicand	$Y_{k-1} Y_{k-2} \ldots Y_1\, Y_0$
X	Multiplier	$x_{m-1} x_{m-2} \ldots x_1\, x_0$
P	Product ($Y \cdot X$)	$p_{m+k-1} p_{m+k-2} \ldots p_2\, p_1\, p_0$

If multiplicand and multiplier are of different sizes, usually multiplier has the smaller size

Xilinx FPGA Implementation Equations

$$Z = (2x_{m-1}+x_{m-2}) \cdot Y \cdot 2^{m-2} + \ldots + (2x_{i+1}+x_i) \cdot Y \cdot 2^i + \ldots +$$
$$+(2x_3+x_2) \cdot Y \cdot 2^2 + (2x_1+x_0) \cdot Y \cdot 2^0$$

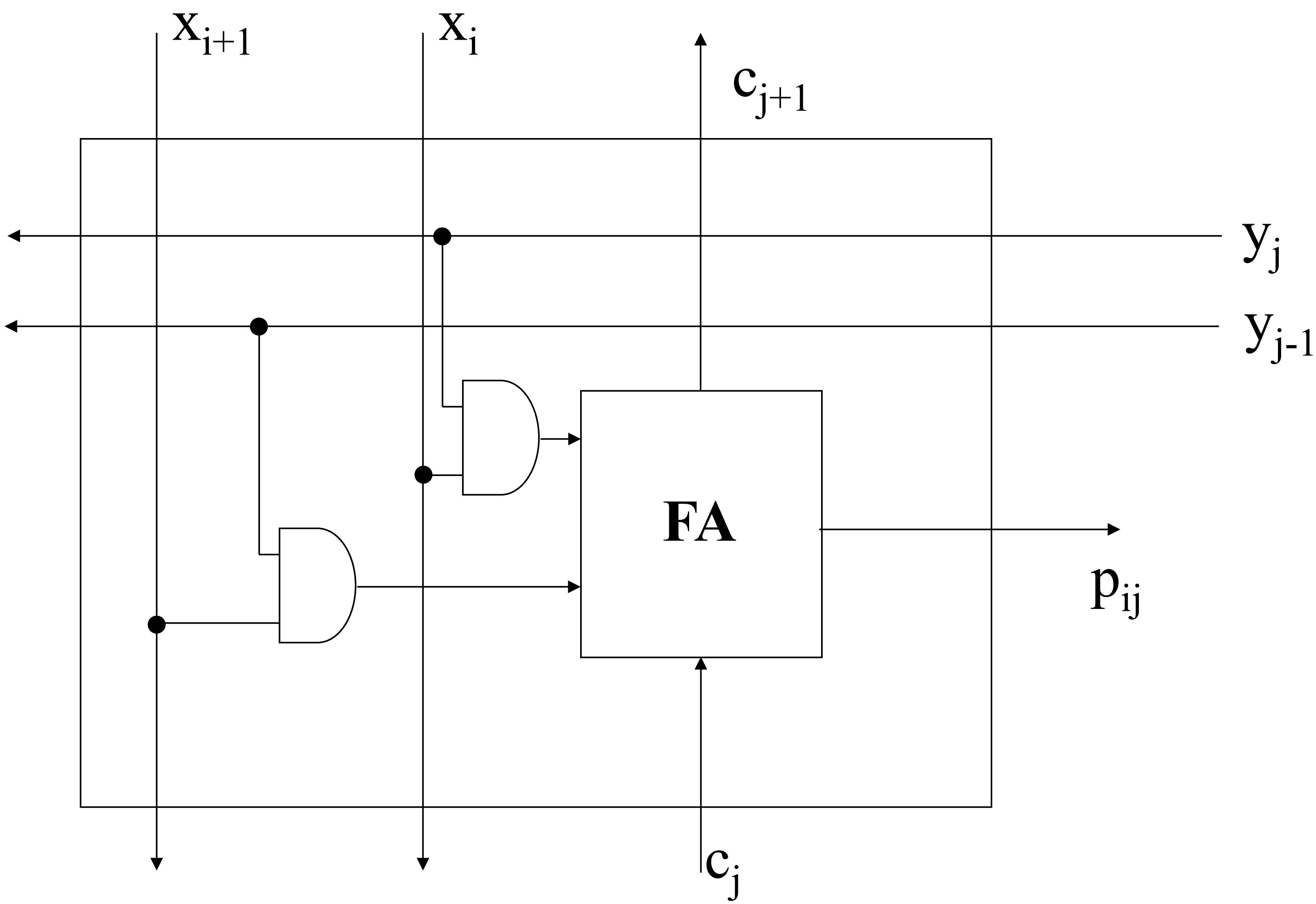
$$(2x_{i+1}+x_i) \cdot Y = p_{i(k+1)}p_{ik}p_{i(k-1)}\ldots p_{i2}p_{i1}p_{i0}$$

$$p_{ij} = x_i \cdot y_j \text{ xor } x_{i+1} \cdot y_{j-1} \text{ xor } c_j$$

$$c_{j+1} = (x_i \cdot y_j)(x_{i+1} \cdot y_{j-1}) + (x_i \cdot y_j) \cdot c_j + (x_{i+1} \cdot y_{j-1}) \cdot c_j$$

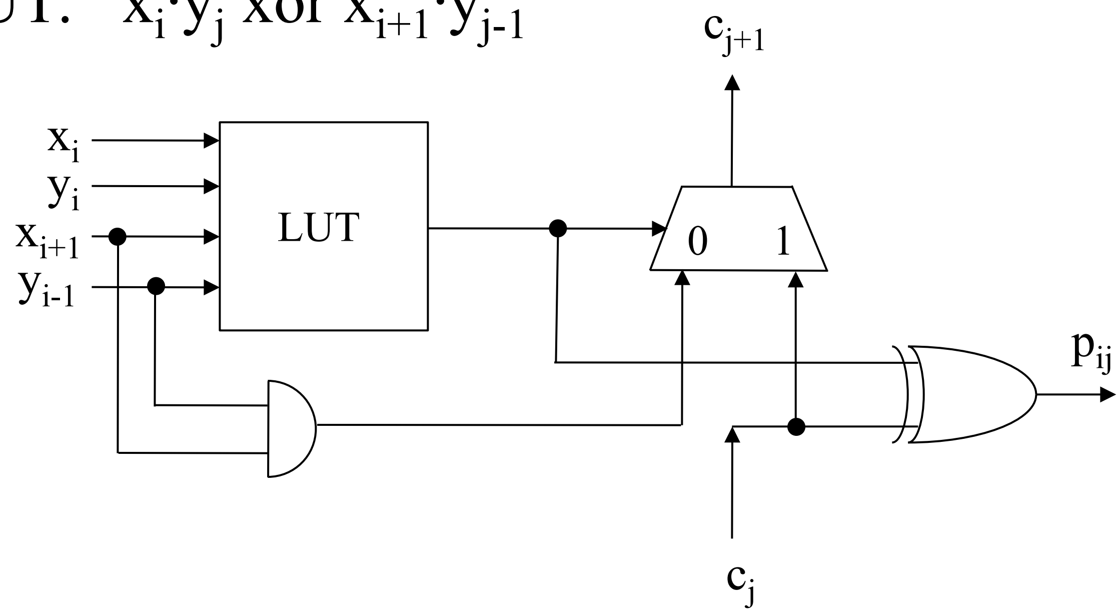
$$c_0 = c_1 = 0$$

Modified Basic Cell
Xilinx FPGA Implementation



Modified Basic Cell
Xilinx FPGA Implementation

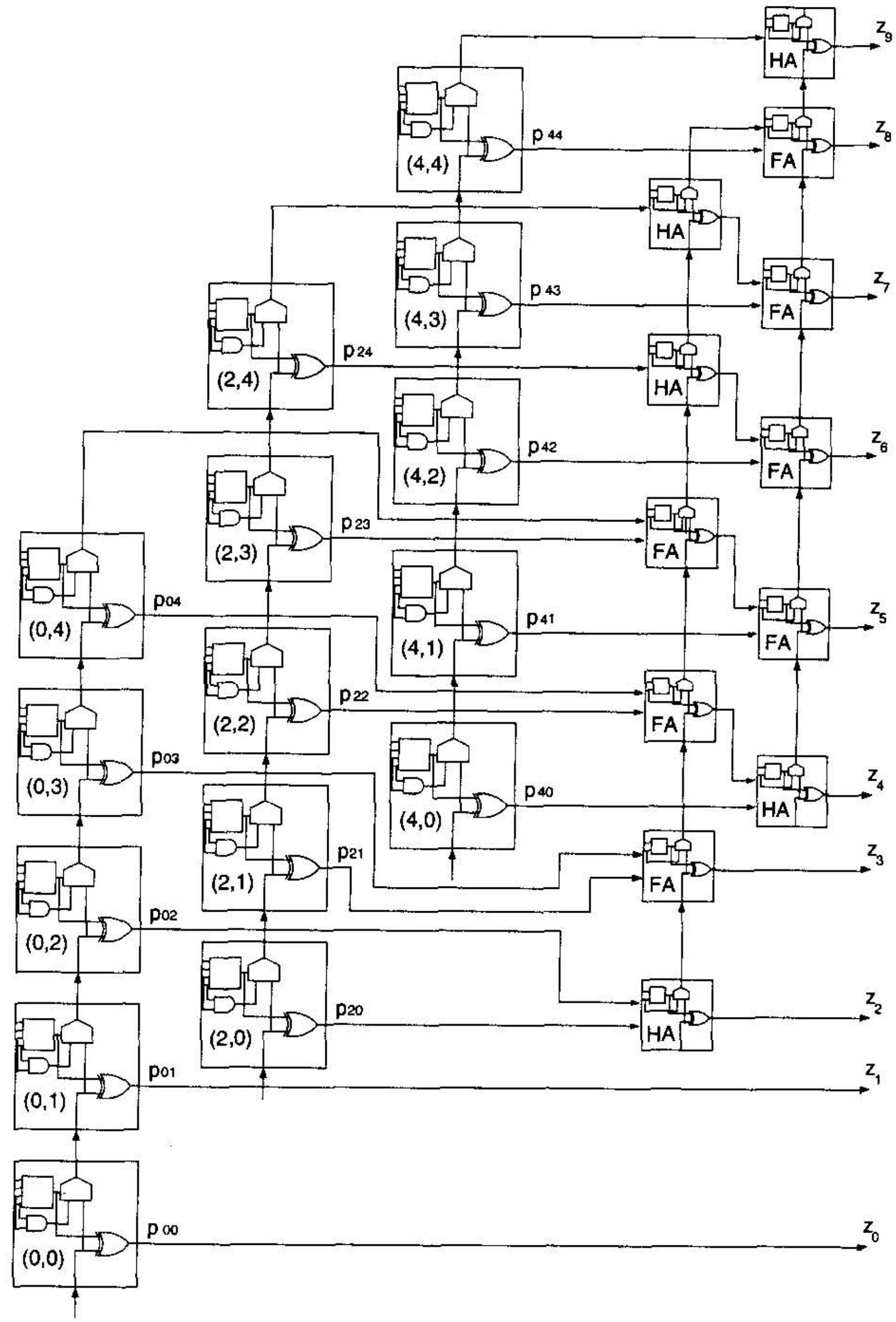
LUT: $x_i \cdot y_j$ xor $x_{i+1} \cdot y_{j-1}$



$$p_{ij} = x_i \cdot y_j \text{ xor } x_{i+1} \cdot y_{j-1} \text{ xor } c_j$$

$$c_{j+1} = (x_i \cdot y_j)(x_{i+1} \cdot y_{j-1}) + (x_i \cdot y_j) \cdot c_j + (x_{i+1} \cdot y_{j-1}) \cdot c_j$$

Xilinx FPGA Multiplier



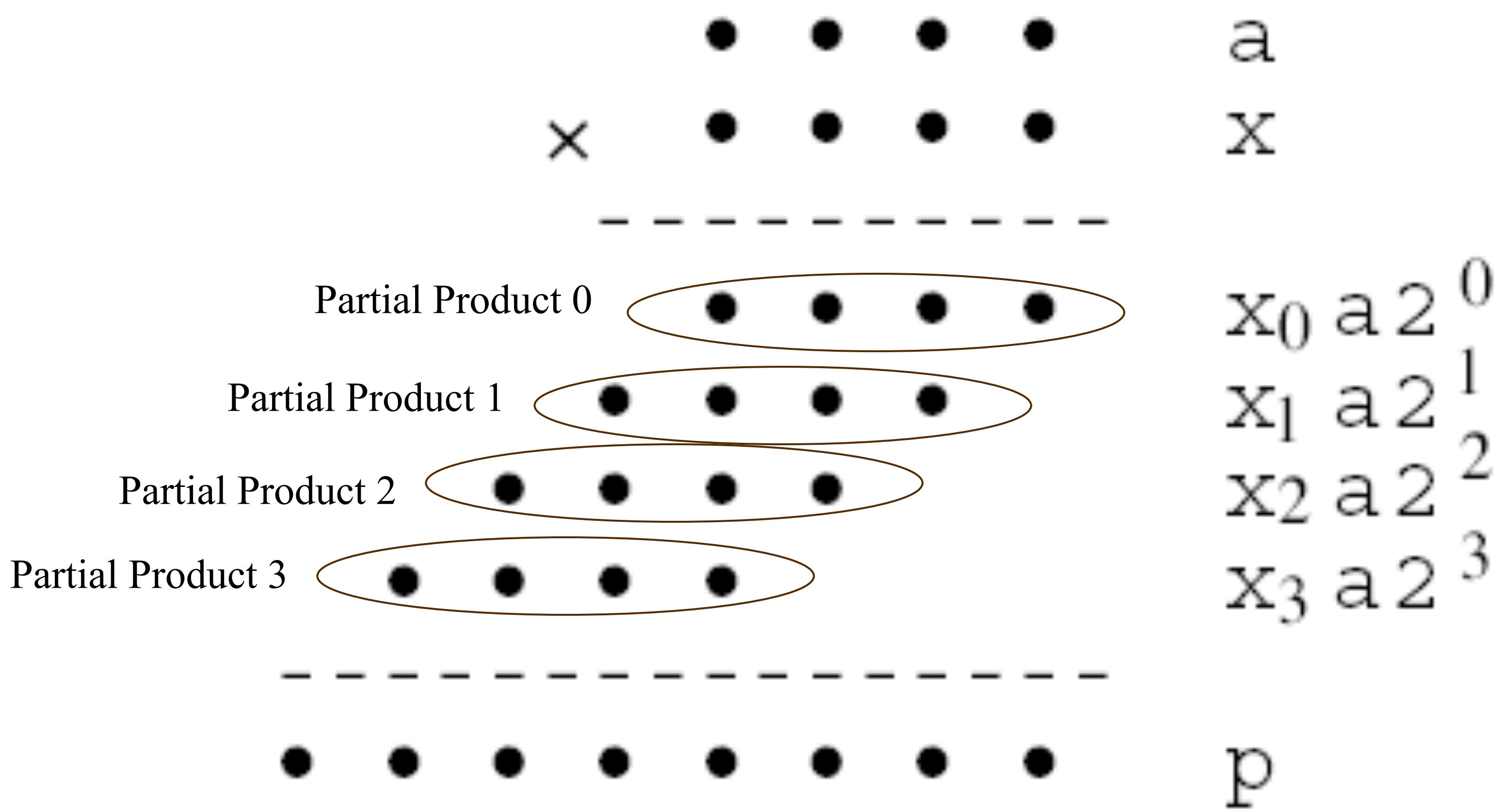
Radix 2
Sequential Multipliers

Notation

a	Multiplicand	$a_{k-1}a_{k-2} \ldots a_1\ a_0$
x	Multiplier	$x_{k-1}x_{k-2} \ldots x_1\ x_0$
p	Product $(a \cdot x)$	$p_{2k-1}p_{2k-2} \ldots p_2\ p_1\ p_0$

If multiplicand and multiplier are of different sizes,
usually multiplier has the smaller size

Multiplication of two 4-bit unsigned binary numbers in dot notation



Number of partial products = number of bits in multiplier x
Bit-width of each partial product = bit-width of multiplicand a

Basic Multiplication Equations

$$p = a \cdot x \qquad\qquad x = \sum_{i=0}^{k-1} x_i \cdot 2^i$$

$$p = a \cdot x = \sum_{i=0}^{k-1} a \cdot x_i \cdot 2^i =$$

$$= x_0 a 2^0 + x_1 a 2^1 + x_2 a 2^2 + \ldots + x_{k-1} a 2^{k-1}$$

Shift/Add Algorithm
Right-shift version

Shift/Add Algorithms
Right-shift algorithm

$$p = a \cdot x = x_0 a 2^0 + x_1 a 2^1 + x_2 a 2^2 + \ldots + x_{k-1} a 2^{k-1} =$$

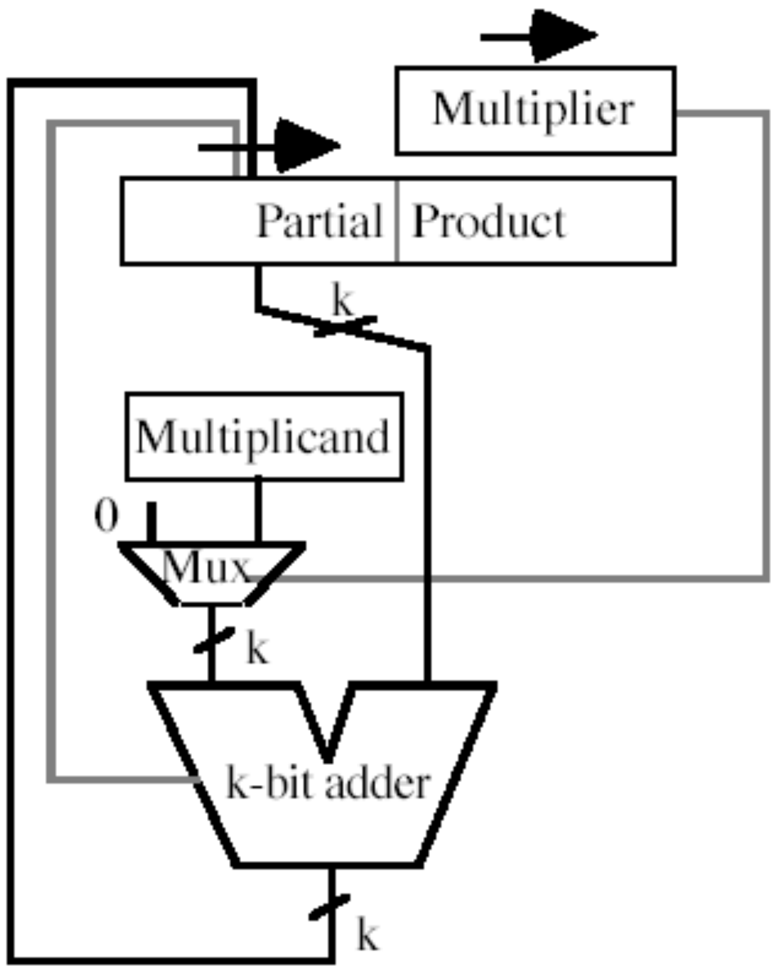
$$= (\ldots((0 + \underbrace{x_0 a 2^k)/2 + x_1 a 2^k)/2 + \ldots + x_{k-1} a 2^k)/2}_{k \text{ times}} =$$

$$p^{(0)} = 0$$

$$p^{(j+1)} = (p^{(j)} + x_j\, a\, 2^k) / 2 \qquad j=0..k-1$$

$$p = p^{(k)}$$

Sequential shift-and-add multiplier for right-shift algorithm

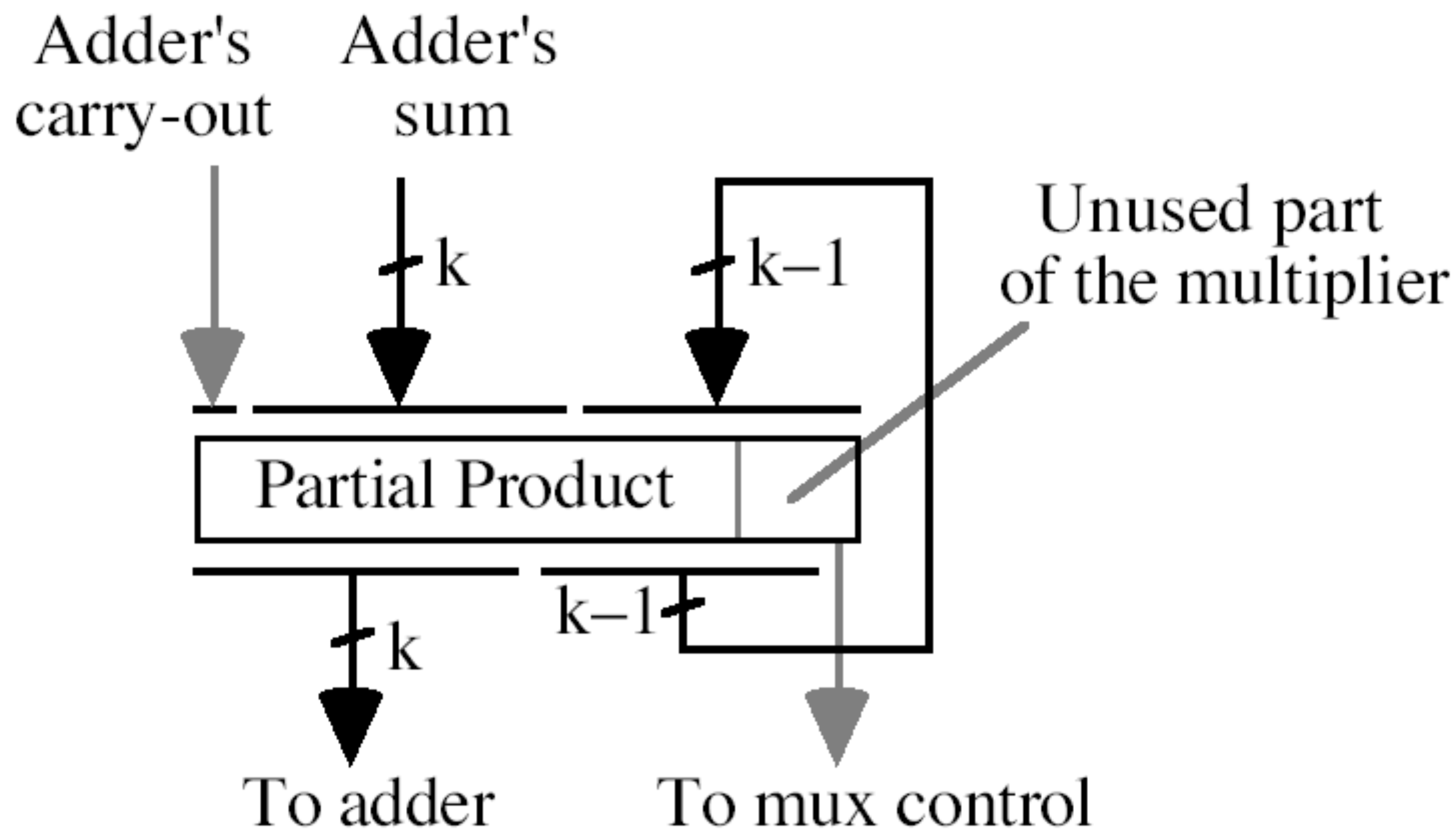


Right-shift multiplication algorithm: Example

Right-shift algorithm

```
=====================
a            1 0 1 0
x            1 0 1 1
=====================
p(0)         0 0 0 0
+x0a         1 0 1 0
---------------------
2p(1)      0 1 0 1 0
p(1)         0 1 0 1   0
+x1a         1 0 1 0
---------------------
2p(2)      0 1 1 1 1   0
p(2)         0 1 1 1   1 0
+x2a         0 0 0 0
---------------------
2p(3)      0 0 1 1 1   1 0
p(3)         0 0 1 1   1 1 0
+x3a         1 0 1 0
---------------------
2p(4)      0 1 1 0 1   1 1 0
p(4)         0 1 1 0   1 1 1 0
=====================
```

Area optimization for the sequential shift-and-add multiplier with the right-shift algorithm



Shift/Add Algorithms
Right-shift algorithm: multiply-add

$$p^{(0)} = y2^k$$

$$p^{(j+1)} = (p^{(j)} + x_j\, a\, 2^k) / 2 \qquad j=0..k-1$$

$$p = p^{(k)}$$

$$= \underbrace{(\ldots((y2^k + x_0a2^k)/2 + x_1a2^k)/2 + \ldots + x_{k-1}a2^k)/2}_{k \text{ times}} =$$

$$= y + x_0a2^0 + x_1a2^1 + x_2a2^2 + \ldots + x_{k-1}a2^{k-1} = y + a \cdot x$$

Signed Multiplication

- Previous sequential multipliers are for unsigned multiplication
- For signed multiplication:
  - assume sign-extended operation for $p^{(j)} + x_j a$
  - if 2's complement multiplier is POSITIVE
    right-shift sequential algorithms (shift-add) will work directly
  - if 2's complement multiplier is NEGATIVE than we must use "negative weight" for x_{k-1} and subtract $x_{k-1}a$ in the last cycle
- Slight increase in area due to control and one-bit sign extension on inputs of adder
  - Unsigned: k bit number + k bit number → k+1 bit number
  - Signed: k+1 bit sign extended number + k+1 bit sign extended number → k+1 bit number

Sequential multiplication of 2’s-complement numbers with right shifts (positive multiplier)

a	1 0 1 1 0	
x	0 1 0 1 1	
$p^{(0)}$	0 0 0 0 0	
$+x_0a$	1 0 1 1 0	
$2p^{(1)}$	1 1 0 1 1 0	
$p^{(1)}$	1 1 0 1 1	0
$+x_1a$	1 0 1 1 0	
$2p^{(2)}$	1 1 0 0 0 1	0
$p^{(2)}$	1 1 0 0 0	1 0
$+x_2a$	0 0 0 0 0	
$2p^{(3)}$	1 1 1 0 0 0	1 0
$p^{(3)}$	1 1 1 0 0	0 1 0
$+x_3a$	1 0 1 1 0	
$2p^{(4)}$	1 1 0 0 1 0	0 1 0
$p^{(4)}$	1 1 0 0 1	0 0 1 0
$+x_4a$	0 0 0 0 0	
$2p^{(5)}$	1 1 1 0 0 1	0 0 1 0
$p^{(5)}$	1 1 1 0 0	1 0 0 1 0

Sequential multiplication of 2’s-complement numbers with right shifts (negative multiplier)

a	1 0 1 1 0	
x	1 0 1 0 1	
$p^{(0)}$	0 0 0 0 0	
$+x_0a$	1 0 1 1 0	
$2p^{(1)}$	1 1 0 1 1 0	
$p^{(1)}$	1 1 0 1 1	0
$+x_1a$	0 0 0 0 0	
$2p^{(2)}$	1 1 1 0 1 1	0
$p^{(2)}$	1 1 1 0 1	1 0
$+x_2a$	1 0 1 1 0	
$2p^{(3)}$	1 1 0 0 1 1	1 0
$p^{(3)}$	1 1 0 0 1	1 1 0
$+x_3a$	0 0 0 0 0	
$2p^{(4)}$	1 1 1 0 0 1	1 1 0
$p^{(4)}$	1 1 1 0 0	1 1 1 0
$+(-x_4a)$	0 1 0 1 0	
$2p^{(5)}$	0 0 0 1 1 0	1 1 1 0
$p^{(5)}$	0 0 0 1 1	0 1 1 1 0

Shift/Add Algorithm
Left-shift version

Shift/Add Algorithms
Left-shift algorithm

$$p = a \cdot x = x_0 a 2^0 + x_1 a 2^1 + x_2 a 2^2 + \ldots + x_{k-1} a 2^{k-1} =$$

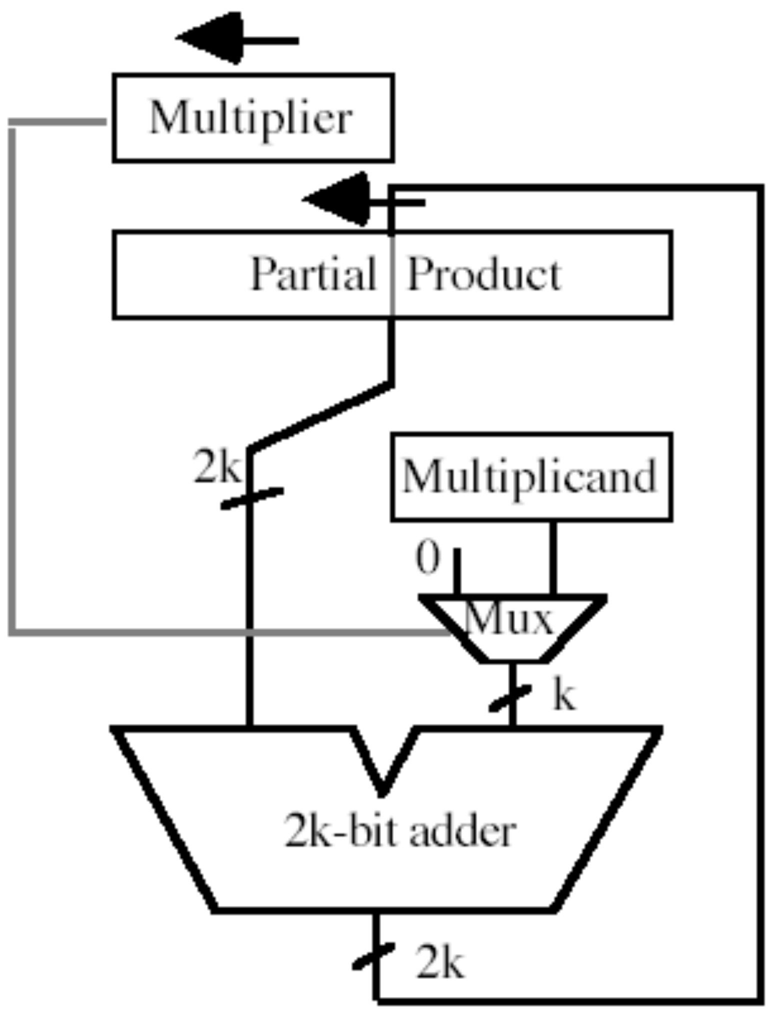
$$= (\ldots((\underbrace{0 \cdot 2 + x_{k-1} a) \cdot 2 + x_{k-2} a) \cdot 2 + \ldots + x_1 a) \cdot 2 + x_0 a}_{k \text{ times}} =$$

$$p^{(0)} = 0$$

$$p^{(j+1)} = (p^{(j)} \cdot 2 + x_{k-1-j} a) \qquad j = 0..k-1$$

$$p = p^{(k)}$$

Sequential shift-and-add multiplier for left-shift algorithm



Left shifts are not as efficient for two's complement because must sign extend multiplicand by k bits

Left-shift multiplication algorithm: Example

Left-shift algorithm

a					1	0	1	0
x					1	0	1	1
$p^{(0)}$					0	0	0	0
$2p^{(0)}$				0	0	0	0	0
$+x_3a$					1	0	1	0
$p^{(1)}$				0	1	0	1	0
$2p^{(1)}$			0	1	0	1	0	0
$+x_2a$					0	0	0	0
$p^{(2)}$			0	1	0	1	0	0
$2p^{(2)}$		0	1	0	1	0	0	0
$+x_1a$					1	0	1	0
$p^{(3)}$		0	1	1	0	0	1	0
$2p^{(3)}$	0	1	1	0	0	1	0	0
$+x_0a$					1	0	1	0
$p^{(4)}$	0	1	1	0	1	1	1	0

Shift/Add Algorithms
Left-shift algorithm: multiply-add

$$p^{(0)} = y2^{-k}$$

$$p^{(j+1)} = (p^{(j)} \cdot 2 + x_{k-(j+1)}a) \qquad j=0..k-1$$

$$p = p^{(k)}$$

$$= (\ldots((y2^{-k} \underbrace{\cdot 2 + x_{k-1}a)\cdot 2 + x_{k-2}a)\cdot 2 + \ldots + x_1 a)\cdot 2 + x_0 a}_{k\ \text{times}} =$$

$$= y + x_{k-1}a2^{k-1} + x_{k-2}a2^{k-2} + \ldots + x_1 a2^1 + x_0 a = y + a \cdot x$$

Shift/Add Algorithm
Right-shift version
with Carry-Save Adder

Sequential shift-and-add multiplier with a carry save adder