The slide features a blue background with a grid of glowing icons representing various computer components and performance metrics. The icons include a CPU, a hard drive, a keyboard, a mouse, a monitor, and a printer. Text elements are overlaid on the background: 'High Performance' in the top left, 'CoolClock' in the top right, 'CoolRunner-II' in the center, 'Low Power' in the bottom left, and 'High Rate' in the bottom right. The main title 'Lecture 10: Fast Dividers' is centered in a bold, dark blue font. Below it, the course name 'ECE 645—Computer Arithmetic' and the date '4/15/08' are displayed in a smaller, dark blue font. A thin blue horizontal line is positioned below the course and date information.

**Lecture 10:
Fast Dividers**

ECE 645—Computer Arithmetic
4/15/08

ECE 645 – Computer Arithmetic

Lecture Roadmap

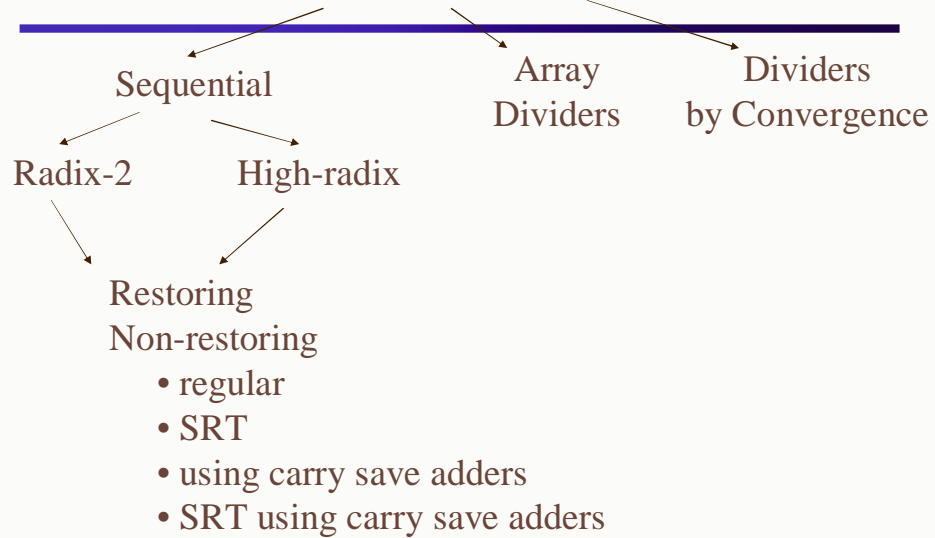
- Array Dividers
- Sequential Dividers Revisited
- SRT Non-Restoring Fractional Division
- Sequential Dividers with Carry-Save Adders
- High-Radix Sequential Dividers
- Multiply-Divide Unit

Required Reading

- B. Parhami, *Computer Arithmetic: Algorithms and Hardware Design*
 - Chapter 14, High-Radix Dividers
 - Chapter 15, Variations in Dividers
- Note errata at:
 - http://www.ece.ucsb.edu/~parhami/text_comp_arit.htm#errors



Classification of Dividers



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Sequential Fractional Division Basic Equations

$$s_{\text{frac}}^{(0)} = z_{\text{frac}}$$

$$s^{(j)} = 2 s^{(j-1)} - q_j d_{\text{frac}}$$

$$s_{\text{frac}}^{(k)} = 2^k s_{\text{frac}}$$

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Restoring Unsigned Fractional Division

$$s^{(0)} = z$$

for $j = 1$ to k

if $2 s^{(j-1)} - d \geq 0$

$$q_j = 1$$

$$s^{(j)} = 2 s^{(j-1)} - d$$

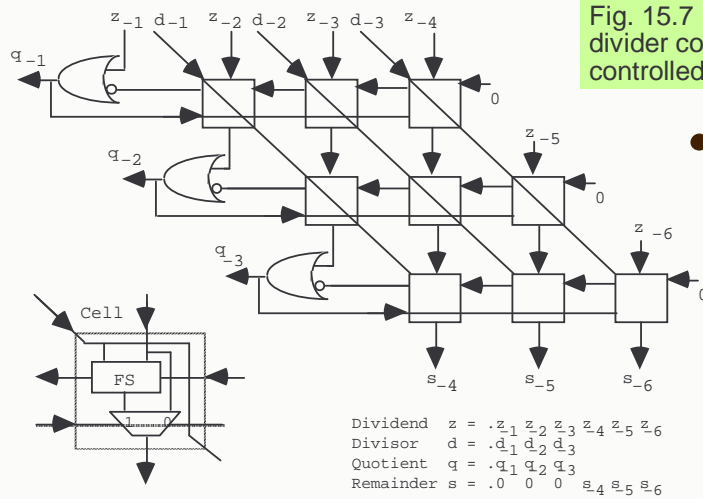
else

$$q_j = 0$$

$$s^{(j)} = 2 s^{(j-1)}$$

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Restoring Array Divider



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Non-Restoring Unsigned Fractional Division

```

 $s^{(-1)} = z - d$ 
for  $j = 0$  to  $k-1$ 
  if  $s^{(j-1)} \geq 0$ 
     $q_j = 1$ 
     $s^{(j)} = 2s^{(j-1)} - d$ 
  else
     $q_j = 0$ 
     $s^{(j)} = 2s^{(j-1)} + d$ 
end for
if  $s^{(k-1)} \geq 0$ 
   $q_k = 1$ 
else
   $q_k = 0$ 
  
```

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Non-Restoring Array Divider

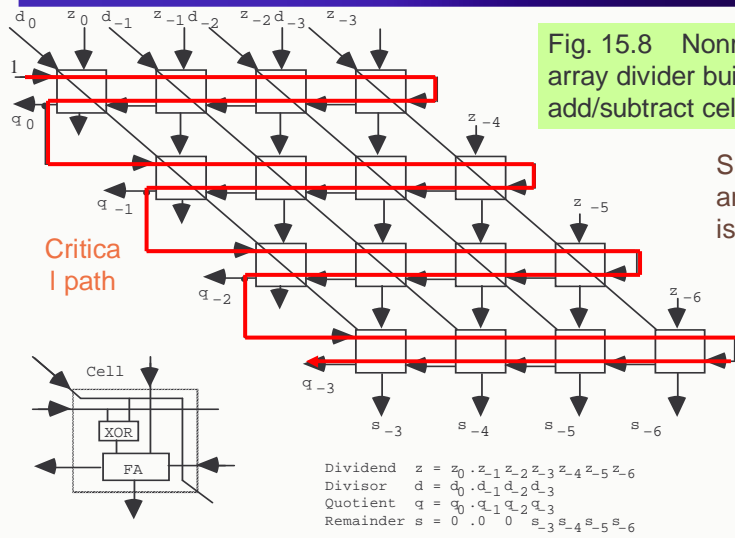


Fig. 15.8 Nonrestoring array divider built of controlled add/subtract cells.

Similarity to array multiplier is deceiving

Sequential Dividers

CoolRunner-II

ECE 645 – Computer Arithmetic

Sequential Fractional Division Basic Equations

$$s_{\text{frac}}^{(0)} = z_{\text{frac}}$$

$$s^{(j)} = 2 s^{(j-1)} - q_{-j} d_{\text{frac}}$$

$$s_{\text{frac}}^{(k)} = 2^k s_{\text{frac}}$$

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Non-Restoring Fractional Division

$$s^{(0)} = z$$

for $j = 1$ to k

if $2s^{(j-1)} \geq 0$

$$q_{-j} = 1$$

$$s^{(j)} = 2 s^{(j-1)} - d$$

else

$$q_{-j} = -1$$

$$s^{(j)} = 2 s^{(j-1)} + d$$

end for

correction step

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Integer Division Correction Step

$$z = q d + s$$

$$\begin{aligned} z &= (q-1) d + (s+d) \\ z &= q' d + s' \end{aligned}$$

$$\begin{aligned} z &= (q+1) d + (s-d) \\ z &= q'' d + s'' \end{aligned}$$

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Fractional Division Correction Step

$$z_{\text{frac}} = q_{\text{frac}} d_{\text{frac}} + s_{\text{frac}}$$

$$\begin{aligned} z_{\text{frac}} &= (q_{\text{frac}} - 2^{-k}) d_{\text{frac}} + (s_{\text{frac}} + d_{\text{frac}} 2^{-k}) \\ z_{\text{frac}} &= q'_{\text{frac}} d_{\text{frac}} + s'_{\text{frac}} \end{aligned}$$

$$\begin{aligned} z_{\text{frac}} &= (q_{\text{frac}} + 2^{-k}) d_{\text{frac}} + (s_{\text{frac}} - d_{\text{frac}} 2^{-k}) \\ z_{\text{frac}} &= q''_{\text{frac}} d_{\text{frac}} + s''_{\text{frac}} \end{aligned}$$

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Partial Remainder in Radix-2 Non-Restoring Division

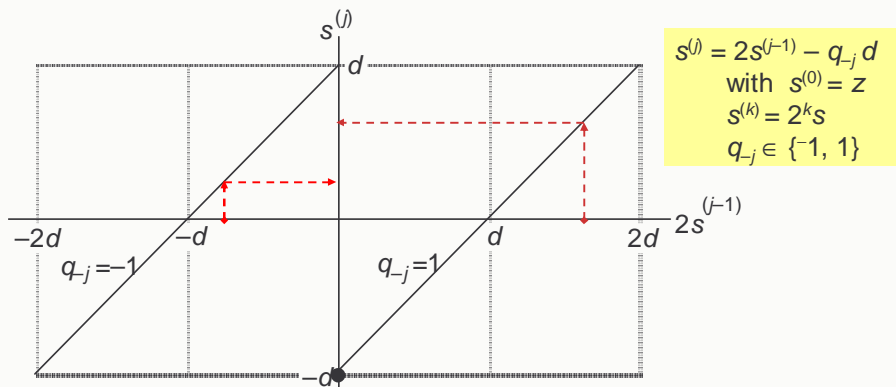


Fig. 14.3 The new partial remainder, $s^{(j)}$, as a function of the shifted old partial remainder, $2s^{(j-1)}$, in radix-2 nonrestoring division.

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Partial Remainder with q_j in $\{-1, 0, 1\}$

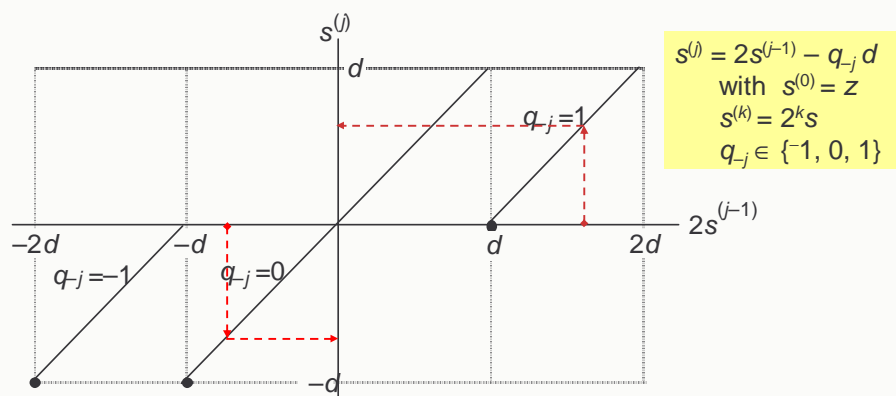


Fig. 14.4 The new partial remainder, $s^{(j)}$, as a function of the shifted old partial remainder, $2s^{(j-1)}$, with q_{-j} in $\{-1, 0, 1\}$.

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Non-Restoring Fractional Division with Shifting Over Zeros

```
s(0) = z
for j = 1 to k
  if 2s(j-1) ≥ d
    qj = 1
    s(j) = 2 s(j-1) - d
  elseif 2s(j-1) < -d
    qj = -1
    s(j) = 2 s(j-1) + d
  else
    qj = 0
    s(j) = 2 s(j-1)
  end for
correction step
```

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High Performance

DataLock

Low Power

DataGate

CoolRunner-II

SRT Non-Restoring Fractional Dividers

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SRT Non-Restoring Fractional Division Assumptions

$d \geq 1/2$ (positive, bit-normalized divider)

$-d \leq -1/2 \leq z, s^{(j)} < 1/2 < d$

If the latter condition not true:

$z = z \gg 1$

perform $k+1$ instead of k steps of the algorithm

$q = q \ll 1$ and $s = s \ll 1$

$z' = z/2$ and $z' = q' \cdot d + s'$



$z = 2z' = (2q') \cdot d + 2s' = q \cdot d + s$

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New and Old Partial Remainders in SRT Division

We use the comparison constants $-1/2$ and $1/2$ for quotient digit selection

$2s \geq +1/2$ means $2s = (0.1xxxxxxx)_{2^s\text{-compl}}$

$2s < -1/2$ means $2s = (1.0xxxxxxx)_{2^s\text{-compl}}$

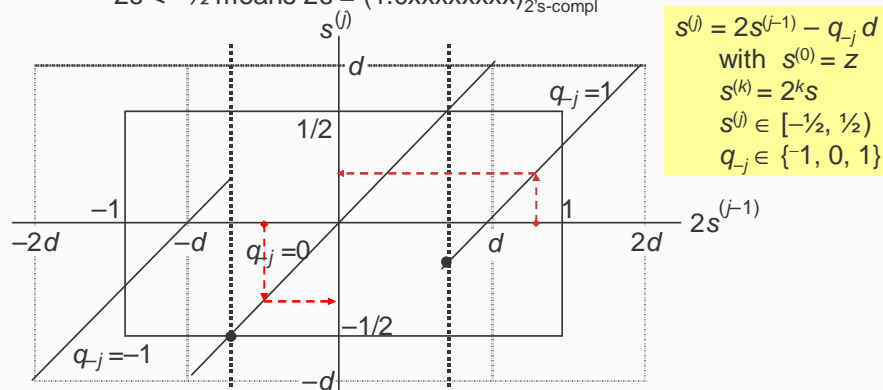


Fig. 14.5 The relationship between new and old partial remainders in radix-2 SRT division.

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SRT Non-Restoring Fractional Division

```
s(0) = z
for j = 1 to k
  if 2s(j-1) ≥ 1/2
    qj = 1
    s(j) = 2 s(j-1) - d
  elseif 2s(j-1) < -1/2
    qj = -1
    s(j) = 2 s(j-1) + d
  else
    qj = 0
    s(j) = 2 s(j-1)
  end for
correction steps
```

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SRT Partial Remainder

SRT algorithm (Sweeney, Robertson, Tocher)

$$2s^{(j-1)} \geq +1/2 = (0.1)_{2\text{'s-compl}}$$
$$\Rightarrow 2s^{(j-1)} = (0.1u_{-2}u_{-3} \dots)_{2\text{'s-compl}}$$

$$2s^{(j-1)} < -1/2 = (1.1)_{2\text{'s-compl}}$$
$$\Rightarrow 2s^{(j-1)} = (1.0u_{-2}u_{-3} \dots)_{2\text{'s-compl}}$$

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Skipping by Shifting

Skipping over identical leading bits by shifting

$s^{(j-1)} = 0.0000110 \dots$ Shift left by 4 bits and subtract;
append q with 0 0 0 1

$s^{(j-1)} = 1.1110100 \dots$ Shift left by 3 bits and add;
append q with 0 0⁻1

Average skipping distance in variable-shift SRT division:
2.67 bits (obtained by statistical analysis)

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SRT Algorithm Observations

We use the comparison constants $-\frac{1}{2}$ and $\frac{1}{2}$ for quotient digit selection

For $2s \geq +\frac{1}{2}$ or $2s = (0.1xxxxxxx)_{2^s\text{-compl}}$ choose $q_{-j} = 1$
For $2s < -\frac{1}{2}$ or $2s = (1.0xxxxxxx)_{2^s\text{-compl}}$ choose $q_{-j} = -1$

Choose $q_{-j} = 0$ in other cases, that is, for:

$0 \leq 2s < +\frac{1}{2}$ or $2s = (0.0xxxxxxx)_{2^s\text{-compl}}$
 $-\frac{1}{2} \leq 2s < 0$ or $2s = (1.1xxxxxxx)_{2^s\text{-compl}}$

Observation: What happens when the magnitude of $2s$ is fairly small?

$2s = (0.00001xxxx)_{2^s\text{-compl}}$ Choosing $q_{-j} = 0$ would lead to the
same condition in the next step;
generate 5 quotient digits 0 0 0 0 1

$2s = (1.1110xxxxx)_{2^s\text{-compl}}$ Generate 4 quotient digits 0 0 0⁻1

Use leading 0s or leading 1s detection circuit to determine how many
quotient digits can be spewed out at once

Statistically, the average skipping distance will be 2.67 bits

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Unsigned Radix-2 SRT Division

		In $[-\frac{1}{2}, \frac{1}{2}]$, so okay ←	
z	.0100 0101		Example Unsigned Radix-2 SRT Division
d	0.1010		
$-d$	1.0110		
<hr/>			
$s^{(0)}$	0.0100 0101		
$2s^{(0)}$	0.1000 101	$\geq \frac{1}{2}$, so set $q_{-1} = 1$	
$+(-d)$	1.0110	and subtract	
<hr/>			
$s^{(1)}$	1.1110 101		
$2s^{(1)}$	1.1101 01	In $[-\frac{1}{2}, \frac{1}{2}]$, so set $q_{-2} = 0$	
<hr/>			
$s^{(2)} = 2s^{(1)}$	1.1101 01		
$2s^{(2)}$	1.1010 1	In $[-\frac{1}{2}, \frac{1}{2}]$, so set $q_{-3} = 0$	
<hr/>			
$s^{(3)} = 2s^{(2)}$	0.1010 1		
$2s^{(3)}$	1.0101	$< -\frac{1}{2}$, so set $q_{-4} = -1$	
$+d$	0.1010	and add	
<hr/>			
$s^{(4)}$	1.1111	Negative,	
$+d$	0.1010	so add to correct	
<hr/>			
$s^{(4)}$	0.1001		
s	0.0000 0101		
q	0.1001	Uncorrected BSD quotient	
q	0.0110	Convert and subtract <i>ulp</i>	

Fig. 14.6
Example of
unsigned
radix-2 SRT
division.

Sequential Dividers with Carry-Save Adders



Partial Remainder in Stored Carry Form

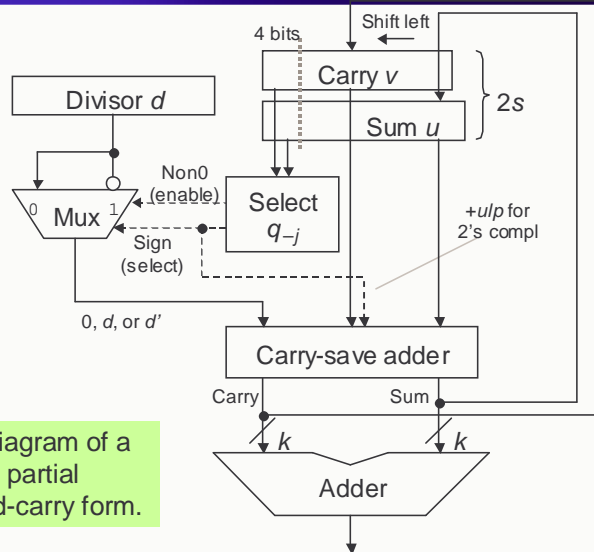


Fig. 14.8 Block diagram of a radix-2 divider with partial remainder in stored-carry form.

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Using Carry Save Adders with Dividers

$$\text{sum} = u = u_1u_0.u_{-1}u_{-2}u_{-3}u_{-4}\dots u_{-k}$$

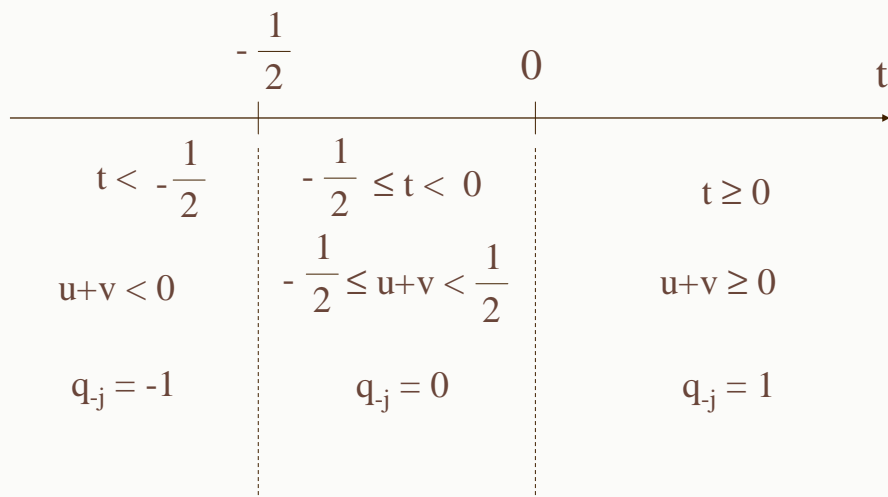
$$\text{carry} = v = v_1v_0.v_{-1}v_{-2}v_{-3}v_{-4}\dots v_{-k}$$

$$t = u_1u_0.u_{-1}u_{-2} + v_1v_0.v_{-1}v_{-2}$$

$$0 \leq \begin{array}{l} u + v - t = 00.00u_{-3}u_{-4}\dots u_{-k} + \\ \quad \quad \quad 00.00v_{-3}v_{-4}\dots v_{-k} \end{array} < \frac{1}{2}$$

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Using Carry-Save Adders with the Dividers



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Sum and Carry

Sum part of $2s^{(j-1)}$: $u = (u_1 u_0 \cdot u_{-1} u_{-2} \cdot \cdot \cdot)_{2\text{'s-compl}}$

Carry part of $2s^{(j-1)}$: $v = (v_1 v_0 \cdot v_{-1} v_{-2} \cdot \cdot \cdot)_{2\text{'s-compl}}$

$t = u_{[-2,1]} + v_{[-2,1]}$ {Add the 4 MSBs of u and v}

if $t < -1/2$

then $q_{-j} = -1$

else if $t \geq 0$

then $q_{-j} = 1$

else $q_{-j} = 0$

endif

endif

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Constant Thresholds Used for Quotient Digit Selection in Radix-2 Division with q_{k-j} in $\{-1, 0, 1\}$

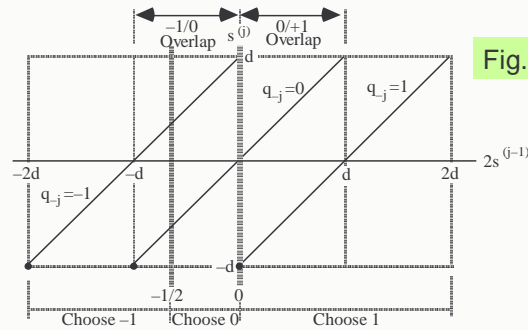


Fig. 14.7

```

t := u_{[-2,1]} + v_{[-2,1]}
if t < -1/2
then q_j = -1
else if t ≥ 0
then q_j = 1
else q_j = 0
endif
endif
    
```

Sum part of $2s^{(j-1)}$: $u = (u_1 u_0 \cdot u_{-1} u_{-2} \cdot \cdot \cdot)_{2^s\text{-compl}}$
 Carry part of $2s^{(j-1)}$: $v = (v_1 v_0 \cdot v_{-1} v_{-2} \cdot \cdot \cdot)_{2^s\text{-compl}}$

Approximation to the partial remainder:

$$t = u_{[-2,1]} + v_{[-2,1]} \quad \{\text{Add the 4 MSBs of } u \text{ and } v\}$$

Max error in approximation

$$< 1/4 + 1/4 = 1/2$$

Error in $[0, 1/2]$

p-d Plot for Radix-2 Division

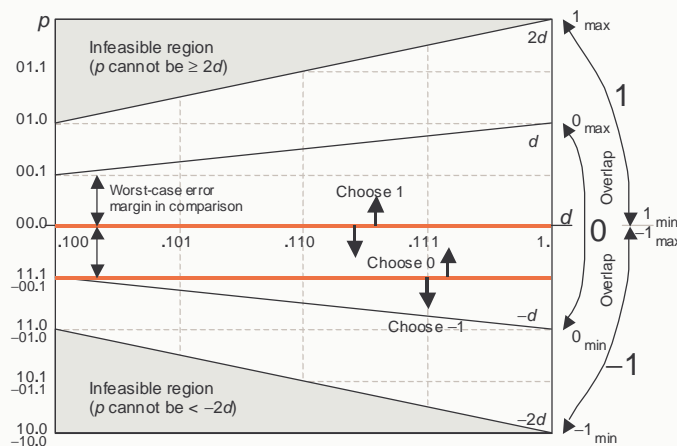


Fig. 14.10 A p - d plot for radix-2 division with $d \in [1/2, 1)$, partial remainder in $[-d, d)$, and quotient digits in $[-1, 1]$.

New Versus Shifted Old Partial Remainder in Radix-4 Division

Radix-4 fractional division with left shifts and $q_{-j} \in [-3, 3]$

$$s^{(j)} = 4s^{(j-1)} - q_{-j}d \quad \text{with } s^{(0)} = z \text{ and } s^{(k)} = 4^k s$$

[-shift-]
[-subtract-]

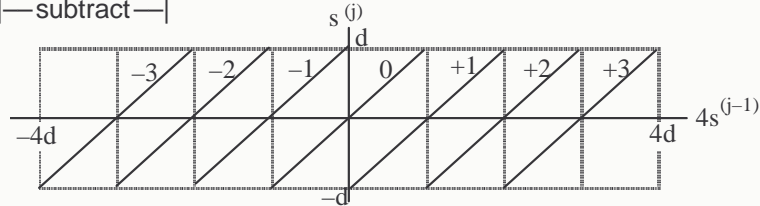


Fig. 14.11 New versus shifted old partial remainder in radix-4 division with q_{-j} in $[-3, 3]$.

Two difficulties:

How do you choose from among the 7 possible values for q_{-j} ?
If the choice is +3 or -3, how do you form $3d$?

p-d Plot for Radix-4 SRT Division with Digit Set [-3,3]

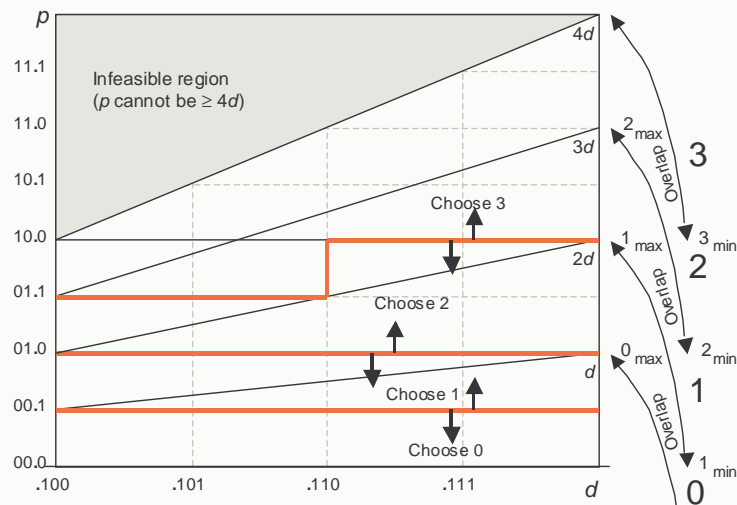


Fig. 14.12 A p - d plot for radix-4 SRT division with quotient digit set $[-3, 3]$.

Radix-4 SRT Divider with the Digit Set $\{-2, -1, 0, 1, 2\}$

Radix-4 fractional division with left shifts and $q_j \in [-2, 2]$

$$s^{(j)} = 4s^{(j-1)} - q_j d \quad \text{with } s^{(0)} = z \text{ and } s^{(k)} = 4^k s$$

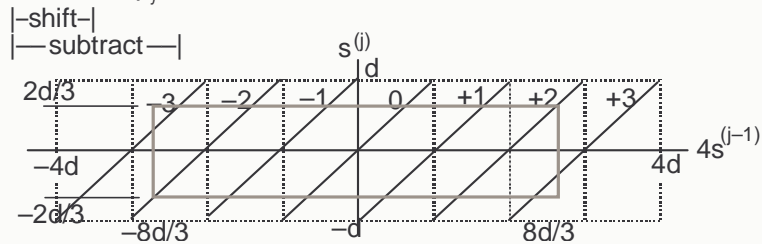


Fig. 14.13 New versus shifted old partial remainder in radix-4 division with q_j in $[-2, 2]$.

For this restriction to be feasible, we must have:

$$s \in [-hd, hd) \text{ for some } h < 1, \text{ and } 4hd - 2d \leq hd$$

This yields $h \leq 2/3$ (choose $h = 2/3$ to minimize the restriction)

p - d Plot for Radix-4 SRT Division with Digit Set $[-2, 2]$

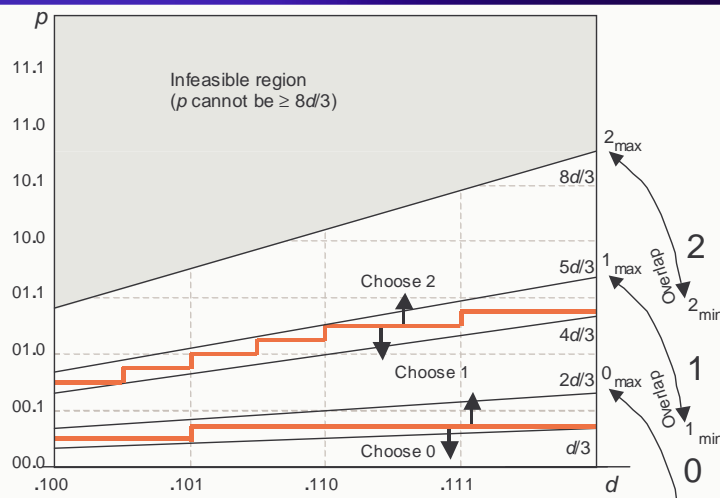
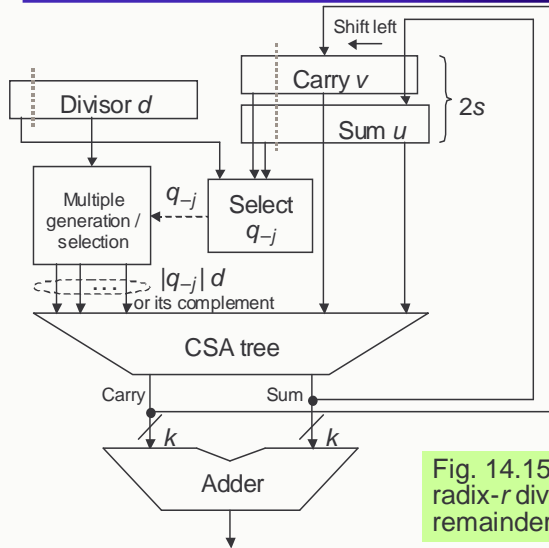


Fig. 14.14 A p - d plot for radix-4 SRT division with quotient digit set $[-2, 2]$.

Radix r Divider



Process to derive the details:

Radix r

Digit set $[-\alpha, \alpha]$ for q_j

Number of bits of p (v and u) and d to be inspected

Quotient digit selection unit (table or logic)

Multiple generation/selection scheme

Conversion of redundant q to 2's complement

Fig. 14.15 Block diagram of radix- r divider with partial remainder in stored-carry form.

High Performance
CoolClock
Low Power
DataSafe

CoolRunner-II

Multiply-Divide Unit

ECE 645 - Computer Arithmetic

Multiply-Divide Unit

The control unit proceeds through necessary steps for multiplication or division (including using the appropriate shift direction)

The slight speed penalty owing to a more complex control unit is insignificant

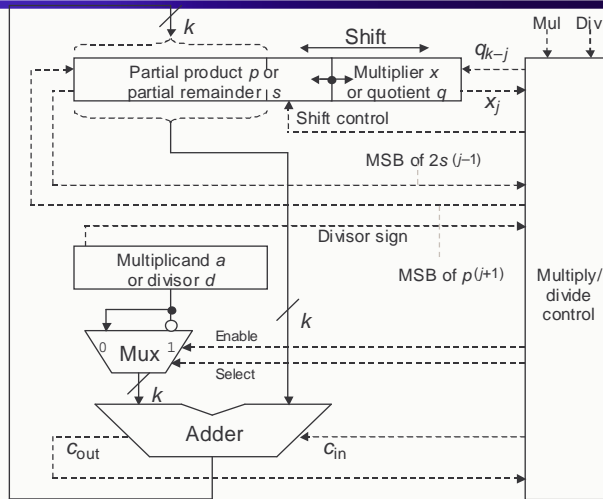


Fig. 15.9 Sequential radix-2 multiply/divide unit.