The slide features a blue background with a grid pattern. Various computer-related icons are scattered across the background, including a monitor, a keyboard, a mouse, a printer, and a scanner. The text "High Performance" is written in a large, stylized font on the left, and "CoolClock" is on the right. In the center, the text "CoolRunner-II" is visible above a central image of a chip. The main title "Lecture 9: Basic Dividers" is prominently displayed in the center. Below it, the course name "ECE 645—Computer Arithmetic" and the date "4/1/08" are listed. At the bottom left, the text "ECE 645 – Computer Arithmetic" is written in a smaller font.

**Lecture 9:  
Basic Dividers**

ECE 645—Computer Arithmetic  
4/1/08

*ECE 645 – Computer Arithmetic*

## Lecture Roadmap

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- Division Notation
- Sequential Division
  - Sequential Integer Unsigned Division
  - Sequential Fractional Unsigned Division
- Restoring Division
  - Restoring Unsigned Division
  - Restoring Signed Division
- Non-Restoring Division
  - Non-Restoring Unsigned Division
  - Non-Restoring Signed Division

## Required Reading

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- B. Parhami, *Computer Arithmetic: Algorithms and Hardware Design*
  - Chapter 13, Basic Division Schemes
- Note errata at:
  - [http://www.ece.ucsb.edu/~parhami/text\\_comp\\_arit.htm#errors](http://www.ece.ucsb.edu/~parhami/text_comp_arit.htm#errors)



## Notation

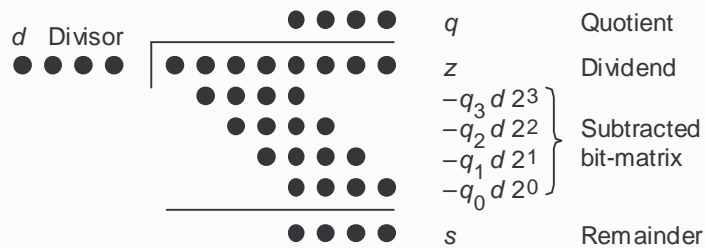
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$z$	Dividend	$z_{2k-1} z_{2k-2} \dots z_2 z_1 z_0$
$d$	Divisor	$d_{k-1} d_{k-2} \dots d_1 d_0$
$q$	Quotient	$q_{k-1} q_{k-2} \dots q_1 q_0$
$s$	Remainder ( $s = z - dq$ )	$s_{k-1} s_{k-2} \dots s_1 s_0$

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## Division of 8-bit number by 4-bit number in dot notation

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## Basic Equations of Division

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$$z = d q + s$$

$$|s| < |d|$$

$$\text{sign}(s) = \text{sign}(z)$$

$$z > 0 \\ 0 \leq s < |d|$$

$$z < 0 \\ -|d| < s \leq 0$$

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## Examples of Proper Signs to Fulfill Basic Division Equation

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Examples of division with signed operands

$$z = 5 \quad d = 3 \quad \Rightarrow \quad q = 1 \quad s = 2$$

$$z = 5 \quad d = -3 \quad \Rightarrow \quad q = -1 \quad s = 2$$

$$z = -5 \quad d = 3 \quad \Rightarrow \quad q = -1 \quad s = -2$$

$$z = -5 \quad d = -3 \quad \Rightarrow \quad q = 1 \quad s = -2$$

Magnitudes of  $q$  and  $s$  are unaffected by input signs  
Signs of  $q$  and  $s$  are derivable from signs of  $z$  and  $d$

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## Unsigned Integer Division Overflow

- Must check overflow because obviously the quotient  $q$  can also be  $2k$  bits.
  - For example, if the divisor  $d$  is 1, then the quotient  $q$  is the dividend  $z$ , which is  $2k$  bits
- Condition for no overflow (i.e.  $q$  fits in  $k$  bits):

$$z = q d + s < (2^k - 1) d + d = d 2^k$$

$$z = z_H 2^k + z_L < d 2^k$$

$$z_H < d$$



## Sequential Integer Unsigned Division Basic Equations

- Division with left shifts
  - There is no corresponding version with right shifts

$$s^{(0)} = z$$

$$s^{(j)} = 2 s^{(j-1)} - q_{k-j} (2^k d)$$

$$s^{(k)} = 2^k s$$

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## Sequential Integer Division Justification

$$s^{(1)} = 2 z - q_{k-1} (2^k d)$$

$$s^{(2)} = 2(2 z - q_{k-1} (2^k d)) - q_{k-2} (2^k d)$$

$$s^{(3)} = 2(2(2 z - q_{k-1} (2^k d)) - q_{k-2} (2^k d)) - q_{k-3} (2^k d)$$

.....

$$s^{(k)} = 2(\dots 2(2 z - q_{k-1} (2^k d)) - q_{k-2} (2^k d)) - q_{k-3} (2^k d) \dots$$

$$- q_0 (2^k d) =$$

$$= 2^k z - (2^k d) (q_{k-1} 2^{k-1} + q_{k-2} 2^{k-2} + q_{k-3} 2^{k-3} + \dots + q_0 2^0) =$$

$$= 2^k z - (2^k d) q = 2^k (z - d q) = 2^k s$$

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## Examples of Sequential Division with Integer and Fractional Operands

Integer division		Fractional division	
$z$	0 1 1 1 0 1 0 1	$z_{frac}$	. 0 1 1 1 0 1 0 1
$2^4d$	1 0 1 0	$d_{frac}$	. 1 0 1 0
$s^{(0)}$	0 1 1 1 0 1 0 1	$s^{(0)}$	. 0 1 1 1 0 1 0 1
$2s^{(0)}$	0 1 1 1 0 1 0 1	$2s^{(0)}$	0 . 1 1 1 0 1 0 1
$-q_3 2^4d$	1 0 1 0 $\{q_3=1\}$	$-q_{-1}d$	. 1 0 1 0 $\{q_{-1}=1\}$
$s^{(1)}$	0 1 0 0 1 0 1	$s^{(1)}$	. 0 1 0 0 1 0 1
$2s^{(1)}$	0 1 0 0 1 0 1	$2s^{(1)}$	0 . 1 0 0 1 0 1
$-q_2 2^4d$	0 0 0 0 $\{q_2=0\}$	$-q_{-2}d$	. 0 0 0 0 $\{q_{-2}=0\}$
$s^{(2)}$	1 0 0 1 0 1	$s^{(2)}$	. 1 0 0 1 0 1
$2s^{(2)}$	1 0 0 1 0 1	$2s^{(2)}$	1 . 0 0 1 0 1
$-q_1 2^4d$	1 0 1 0 $\{q_1=1\}$	$-q_{-3}d$	. 1 0 1 0 $\{q_{-3}=1\}$
$s^{(3)}$	1 0 0 0 1	$s^{(3)}$	. 1 0 0 0 1
$2s^{(3)}$	1 0 0 0 1	$2s^{(3)}$	1 . 0 0 0 1
$-q_0 2^4d$	1 0 1 0 $\{q_0=1\}$	$-q_{-4}d$	. 1 0 1 0 $\{q_{-4}=1\}$
$s^{(4)}$	0 1 1 1	$s^{(4)}$	. 0 1 1 1
$s$	0 1 1 1	$s_{frac}$	0 . 0 0 0 0 0 1 1 1
$q$	1 0 1 1	$q_{frac}$	. 1 0 1 1

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## Unsigned Fractional Division Notation

$z_{frac}$  Dividend  $.z_{-1}z_{-2} \dots z_{-(2k-1)}z_{-2k}$

$d_{frac}$  Divisor  $.d_{-1}d_{-2} \dots d_{-(k-1)}d_{-k}$

$q_{frac}$  Quotient  $.q_{-1}q_{-2} \dots q_{-(k-1)}q_{-k}$

$s_{frac}$  Remainder  $\underbrace{.000\dots 0}_{k \text{ bits}}s_{-(k+1)} \dots s_{-(2k-1)}s_{-2k}$

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## Integer versus Fractional Division

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For Integers:

$$z = q d + s \quad | \cdot 2^{-2k}$$

$$z 2^{-2k} = (q 2^{-k}) (d 2^{-k}) + s (2^{-2k})$$

For Fractions:

$$z_{\text{frac}} = q_{\text{frac}} d_{\text{frac}} + s_{\text{frac}}$$

where

$$z_{\text{frac}} = z 2^{-2k}$$

$$q_{\text{frac}} = q 2^{-k}$$

$$d_{\text{frac}} = d 2^{-k}$$

$$s_{\text{frac}} = s 2^{-2k}$$

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## Unsigned Fractional Division Overflow

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Condition for no overflow:

$$z_{\text{frac}} < d_{\text{frac}}$$

---

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## Sequential Fractional Unsigned Division Basic Equations

$$s^{(0)} = z_{\text{frac}}$$

$$s^{(j)} = 2 s^{(j-1)} - q_{-j} d_{\text{frac}}$$

$$2^k \cdot s_{\text{frac}} = s^{(k)}$$

$$s_{\text{frac}} = 2^{-k} \cdot s^{(k)}$$

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## Sequential Fractional Division Justification

$$s^{(1)} = 2 z_{\text{frac}} - q_{-1} d_{\text{frac}}$$

$$s^{(2)} = 2(2 z_{\text{frac}} - q_{-1} d_{\text{frac}}) - q_{-2} d_{\text{frac}}$$

$$s^{(3)} = 2(2(2 z_{\text{frac}} - q_{-1} d_{\text{frac}}) - q_{-2} d_{\text{frac}}) - q_{-3} d_{\text{frac}}$$

.....

$$s^{(k)} = 2(\dots 2(2 z_{\text{frac}} - q_{-1} d_{\text{frac}}) - q_{-2} d_{\text{frac}}) - q_{-3} d_{\text{frac}} \dots$$

$$- q_{-k} d_{\text{frac}} =$$

$$= 2^k z_{\text{frac}} - d_{\text{frac}} (q_{-1} 2^{k-1} + q_{-2} 2^{k-2} + q_{-3} 2^{k-3} + \dots + q_{-k} 2^0) =$$

$$= 2^k z_{\text{frac}} - d_{\text{frac}} 2^k (q_{-1} 2^{-1} + q_{-2} 2^{-2} + q_{-3} 2^{-3} + \dots + q_{-k} 2^{-k}) =$$

$$= 2^k z_{\text{frac}} - (2^k d_{\text{frac}}) q_{\text{frac}} = 2^k (z_{\text{frac}} - d_{\text{frac}} q_{\text{frac}}) = 2^k s_{\text{frac}}$$

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## Restoring Unsigned Integer Division

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$$s^{(0)} = z$$

**for**  $j = 1$  to  $k$

**if**  $2 s^{(j-1)} - 2^k d > 0$

$$q_{k-j} = 1$$

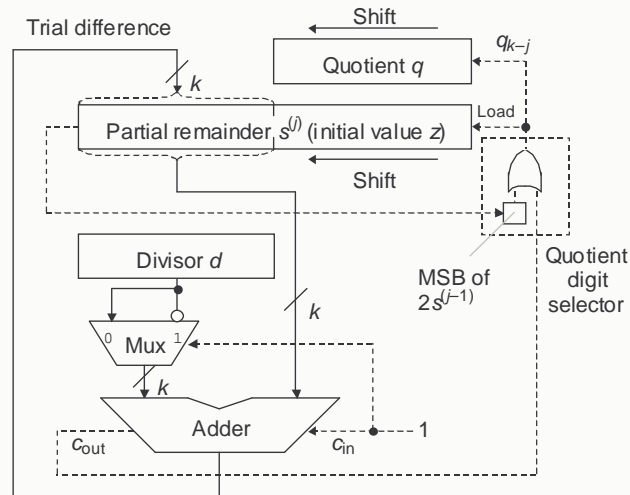
$$s^{(j)} = 2 s^{(j-1)} - 2^k d$$

**else**

$$q_{k-j} = 0$$

$$s^{(j)} = 2 s^{(j-1)}$$

## Shift/Subtract Sequential Restoring Divider



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## Example of Restoring Unsigned Division

=====		
$z$	0 1 1 1 0 1 0 1	No overflow, since:
$2^4 d$	1 0 1 0	$(0111)_{\text{two}} < (1010)_{\text{two}}$
$-2^4 d$	0 1 1 0	
=====		
$s^{(0)}$	0 1 1 1 0 1 0 1	
$2s^{(0)}$	0 1 1 1 0 1 0 1	
$+(-2^4 d)$	0 1 1 0	
$s^{(1)}$	1 ← <sup>cout</sup> 0 1 0 0 1 0 1	<b>cout = 1, so set <math>q_3=1</math></b>
$2s^{(1)}$	0 1 0 0 1 0 1	
$+(-2^4 d)$	0 1 1 0	
$s^{(2)}$	0 ← <sup>cout</sup> 1 1 1 1 0 1	<b>cout = 0, so set <math>q_2=0</math></b>
$s^{(2)}=2s^{(1)}$	1 0 0 1 0 1	<b>and restore</b>
$2s^{(2)}$	1 0 0 1 0 1	
$+(-2^4 d)$	0 1 1 0	
$s^{(3)}$	1 0 0 0 1	<b>MSB of <math>2s^{(2)} = 1</math>, so set <math>q_1=1</math></b>
$2s^{(3)}$	1 0 0 0 1	
$+(-2^4 d)$	0 1 1 0	
$s^{(4)}$	0 1 1 1	<b>MSB of <math>2s^{(3)} = 1</math>, so set <math>q_0=1</math></b>
$s$	0 1 1 1	
$q$	1 0 1 1	
=====		

ALGORITHM:  
 if MSB of  $2s^{(i-1)} = 1$   
 or cout = 1 then set  $q_{k-i} = 1$

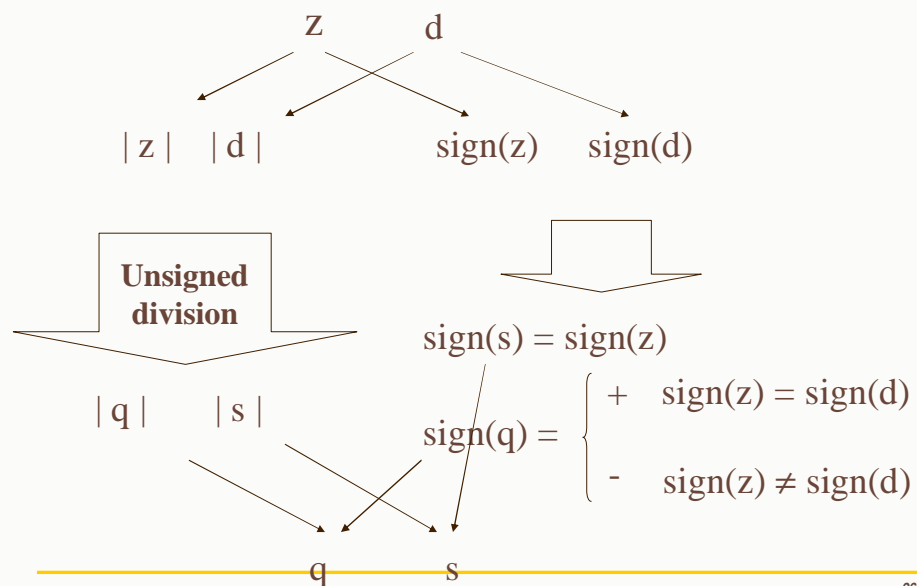
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## Restoring Signed Division

- First convert the divisor and dividend into their equivalent unsigned binary values
- Then apply the correct sign values after the division is finished
- We will see "direct" methods of signed division later

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## Restoring Signed Integer Division



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## Examples

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Examples of division with signed operands

$$z = 5 \quad d = 3 \quad \Rightarrow \quad q = 1 \quad s = 2$$

$$z = 5 \quad d = -3 \quad \Rightarrow \quad q = -1 \quad s = 2$$

$$z = -5 \quad d = 3 \quad \Rightarrow \quad q = -1 \quad s = -2$$

$$z = -5 \quad d = -3 \quad \Rightarrow \quad q = 1 \quad s = -2$$

Magnitudes of  $q$  and  $s$  are unaffected by input signs  
Signs of  $q$  and  $s$  are derivable from signs of  $z$  and  $d$

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**Non-Restoring Division**

*ECE 645 - Computer Arithmetic*

## Non-Restoring and Signed Division

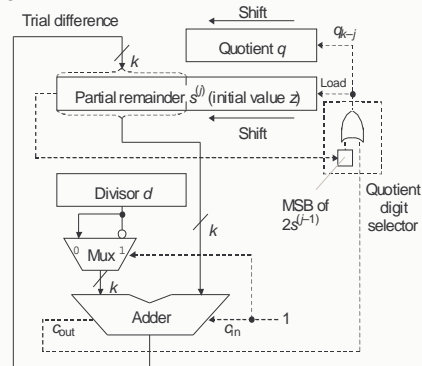
The cycle time in restoring division must accommodate:

- Shifting the registers
- Allowing signals to propagate through the adder
- Determining and storing the next quotient digit
- Storing the trial difference, if required

Later events depend on earlier ones in the same cycle, causing a lengthening of the clock cycle

Nonrestoring division to the rescue!

- Assume  $q_{k-j} = 1$  and subtract
- Store the result as the new PR (the partial remainder can become incorrect, hence the name “nonrestoring”)



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## Justification for Non-Restoring Division

Why it is acceptable to store an incorrect value in the partial-remainder register?

Shifted partial remainder at start of the cycle is  $u$

Suppose subtraction yields the negative result  $u - 2^k d$

Option 1: Restore the partial remainder to correct value  $u$ , shift left, and subtract to get  $2u - 2^k d$

Option 2: Keep the incorrect partial remainder  $u - 2^k d$ , shift left, and add to get  $2(u - 2^k d) + 2^k d = 2u - 2^k d$

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## Non-Restoring Unsigned Division

```

s(1) = 2z - 24d
for j = 2 to k
  if s(j-1) ≥ 0
    qk-(j-1) = 1
    s(j) = 2 s(j-1) - 2k d
  else
    qk-(j-1) = 0
    s(j) = 2 s(j-1) + 2k d
end for
if s(k) ≥ 0
  q0 = 1
else
  q0 = 0
  if s(k) < 0
    Correction step
    for remainder

```

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## Non-Restoring Unsigned Division

```

=====
z          0 1 1 1 0 1 0 1
24d       0 1 0 1 0
-24d      1 0 1 1 0
=====
s(0)       0 0 1 1 1 0 1 0 1
2s(0)     0 1 1 1 0 1 0 1
+(-24d)   1 0 1 1 0
-----
s(1)       0 0 1 0 0 1 0 1
2s(1)     0 1 0 0 1 0 1
+(-24d)   1 0 1 1 0
-----
s(2)       1 1 1 1 1 0 1
2s(2)     1 1 1 1 0 1
+24d      0 1 0 1 0
-----
s(3)       0 1 0 0 0 1
2s(3)     1 0 0 0 1
+(-24d)   1 0 1 1 0
-----
s(4)       0 0 1 1 1
s          0 1 1 1
q          Set q0=1 1 0 1 1
=====

```

No overflow, since:  
 $(0111)_{\text{two}} < (1010)_{\text{two}}$

Always Positive, so subtract

Positive, so set  $q_3 = 1$  and subtract

Negative, so set  $q_2 = 0$  and add

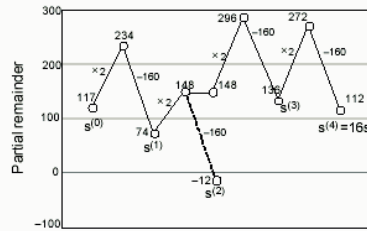
Positive, so set  $q_1 = 1$  and subtract

Positive, so set  $q_0 = 1$

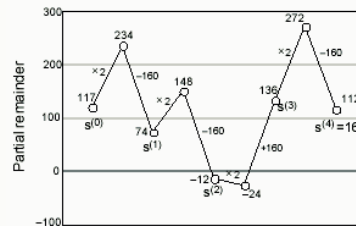
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## Partial Remainder Variations for Restoring and Non-Restoring Division

Example  
 $(01110101)_{\text{two}} / (1010)_{\text{two}}$   
 $(117)_{\text{ten}} / (10)_{\text{ten}}$



(a) Restoring



(b) Nonrestoring

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## Non-Restoring Unsigned Division

### Justification

**Restoring division**

$$s^{(j)} = 2 s^{(j-1)}$$

$$\begin{aligned} s^{(j+1)} &= 2 s^{(j)} - 2^k d = \\ &= 4 s^{(j-1)} - 2^k d \end{aligned}$$

**Non-Restoring division**

$$s^{(j)} = 2 s^{(j-1)} - 2^k d$$

$$\begin{aligned} s^{(j+1)} &= 2 s^{(j)} + 2^k d = \\ &= 2 (2 s^{(j-1)} - 2^k d) + 2^k d = \\ &= 4 s^{(j-1)} - 2^k d \end{aligned}$$

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# Non-Restoring Signed Integer Division

## Correction step

$$z = q d + s$$

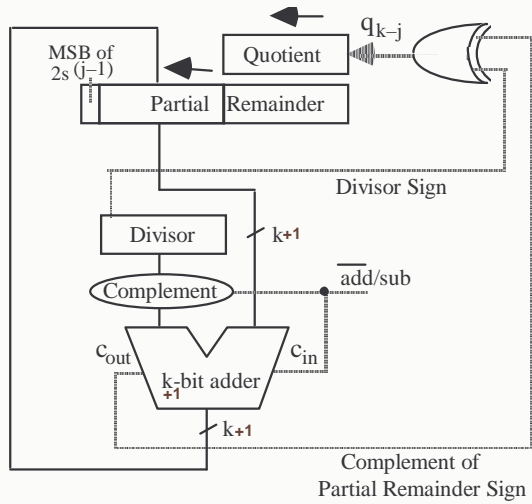
$$z = (q-1) d + (s+d)$$

$$z = q' d + s'$$

$$z = (q+1) d + (s-d)$$

$$z = q'' d + s''$$

# Shift/Subtract Sequential Non-Restoring Divider



**NOTE:**

1.  $q_{k,j}$  and add/subtract control determined by **both** divisor sign and sign of new partial remainder (via an XOR operation). Divisor sign is 0 for unsigned division.
2. note that  $c_{out}$  is the complement of the sign of the new partial remainder, i.e. if new partial remainder = 0, then  $c_{out} = 1$ ; if new partial remainder = 1, then  $c_{out} = 0$

## Non-Restoring Division with Signed Operands

### Restoring division

$q_{k-j} = 0$  means no subtraction (or subtraction of 0)

$q_{k-j} = 1$  means subtraction of  $d$

Example:  $q = \dots 0 0 0 1 \dots$

### Nonrestoring division

$\dots 1 -1 -1 -1 \dots$

We always subtract or add

It is as if quotient digits are selected from the set  $\{1, -1\}$ :

1 corresponds to subtraction       $-1$  corresponds to addition

Our goal is to end up with a remainder that matches the sign of the dividend

This idea of trying to match the sign of  $s$  with the sign  $z$ , leads to a direct signed division algorithm

if  $\text{sign}(s) = \text{sign}(d)$  then  $q_{k-j} = 1$  else  $q_{k-j} = -1$

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## Non-Restoring Signed Integer Division

$s^{(0)} = z$

**for**  $j = 1$  to  $k$

**if**  $\text{sign}(s^{(j-1)}) == \text{sign}(d)$

$q_{k-j} = 1$

$s^{(j)} = 2 s^{(j-1)} - 2^k d = 2 s^{(j-1)} - q_{k-j} (2^k d)$

**else**

$q_{k-j} = -1$

$s^{(j)} = 2 s^{(j-1)} + 2^k d = 2 s^{(j-1)} - q_{k-j} (2^k d)$

Correction\_step

$q = \text{BSD\_2's\_comp\_conversion}(q);$

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## BSD → 2's Complement Conversion

$$q = (q_{k-1} q_{k-2} \dots q_1 q_0)_{\text{BSD}} =$$

$$= (\overline{p_{k-1} p_{k-2} \dots p_1 p_0} 1)_{2\text{'s complement}}$$

where

$q_i$	$p_i$
-1	0
1	1

Example:

$$q_{\text{BSD}} \quad 1 \ -1 \ 1 \ 1$$

$$p \quad 1 \ 0 \ 1 \ 1$$

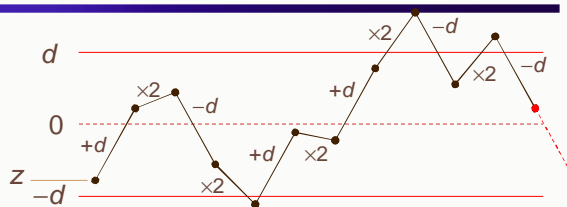
$$q_{2\text{'s comp}} \quad \underline{00} \ 1 \ 1 \ 1 = 0 \ 1 \ 1 \ 1$$

no overflow if  $p_{k-2} = \overline{p_{k-1}}$  ( $q_{k-1} \neq q_{k-2}$ )

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## Quotient Conversion and Final Correction

Partial remainder variation and selected quotient digits during nonrestoring division with  $d > 0$



Quotient with digits -1 and 1

-1 1 -1 -1 1 1

Replace -1s with 0s

0 1 0 0 1 1

Shift left, complement MSB, and set LSB to 1 to get the 2's-complement quotient

1 1 0 0 1 1 1

Check:  $-32 + 16 - 8 - 4 + 2 + 1 = -25 = -64 + 32 + 4 + 2 + 1$

1 1 0 1 0 0 0

Final correction step if  $\text{sign}(s) \neq \text{sign}(z)$ :

Add  $d$  to, or subtract  $d$  from,  $s$ ; subtract 1 from, or add 1 to,  $q$

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### Example of Nonrestoring Signed Division

=====			
$z$		0 0 1 0	0 0 0 1
$2^4d$	1	1 0 0 1	
$-2^4d$	0	0 1 1 1	
=====			
$s^{(0)}$	0	0 0 1 0	0 0 0 1
$2s^{(0)}$	0	0 1 0 0	0 0 1
$+2^4d$	1	1 0 0 1	
=====			
$s^{(1)}$	1	1 1 0 1	0 0 1
$2s^{(1)}$	1	1 0 1 0	0 1
$+(-2^4d)$	0	0 1 1 1	
=====			
$s^{(2)}$	0	0 0 0 1	0 1
$2s^{(2)}$	0	0 0 1 0	1
$+2^4d$	1	1 0 0 1	
=====			
$s^{(3)}$	1	1 0 1 1	1
$2s^{(3)}$	1	0 1 1 1	
$+(-2^4d)$	0	0 1 1 1	
=====			
$s^{(4)}$	1	1 1 1 0	
$+(-2^4d)$	0	0 1 1 1	
=====			
$s^{(4)}$	0	0 1 0 1	
$s$		0 1 0 1	
$q$		-1 1 -1 1	
=====			

$\text{sign}(s^{(0)}) \neq \text{sign}(d)$ ,  
so set  $q_3 = -1$  and add

$\text{sign}(s^{(1)}) = \text{sign}(d)$ ,  
so set  $q_2 = 1$  and subtract

$\text{sign}(s^{(2)}) \neq \text{sign}(d)$ ,  
so set  $q_1 = -1$  and add

$\text{sign}(s^{(3)}) = \text{sign}(d)$ ,  
so set  $q_0 = 1$  and subtract

$\text{sign}(s^{(4)}) \neq \text{sign}(z)$ ,  
so perform corrective subtraction

$p =$  0 1 0 1    Shift, compl MSB  
          1 1 0 1 1    Add 1 to correct  
          1 1 0 0    Check:  $33/(-7) = -4$

Fig. 13.9  
Example of  
nonrestoring  
signed  
division.