Step Response For Type 2 Systems

A. System Model

Assume that the system is modeled as a Single-Input, Single-Output (SISO) configuration with unity feedback and the following forward loop transfer function

\[ G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{s^2 D(s)} \]  

(1)

\[ N(s) \big|_{s=0} \neq 0 \]  

(2)

The expressions in (1) and (2) indicate that the system is Type 2, having two open-loop poles at the origin, and there is no pole-zero cancellation of that term.

With the open-loop system \( G(s) \) defined in (1)–(2), the closed-loop transfer function is

\[ T_{CL}(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{s^2 D(s)}{1 + \frac{N(s)}{s^2 D(s)}} = \frac{N(s)}{\Delta_{CL}(s)} \]  

(3)

It will be assumed that the closed-loop system is internally stable. Therefore, all roots of the closed-loop characteristic equation \( \Delta_{CL}(s) \) lie strictly in the left-half of the complex \( s \)-plane (LHP), and there are no unstable pole-zero cancellations in \( G(s) \)—there are no unstable hidden modes in the system.

The reference input to the system will be the unit step function, so the transform of the reference input is

\[ R(s) = \frac{1}{s} \]  

(4)

The transform of the system output is the product of the transform of the reference input signal and the closed-loop transfer function. Therefore, the transform of the output signal is

\[ C(s) = T_{CL}(s) R(s) = \frac{N(s)}{s \Delta_{CL}(s)} \]  

(5)

The error in the system is defined to be the difference between the reference input and the actual output. The transform for the error signal is

\[ E(s) = R(s) - C(s) = R(s) - T_{CL}(s) R(s) = [1 - T_{CL}(s)] R(s) = \left[ \frac{1}{1 + G(s)} \right] R(s) \]  

(6)

B. Region of Convergence

The single-sided (unilateral) Laplace transform for a signal \( x(t) \) is defined to be

\[ \mathcal{L} [x(t)] = X(s) = \int_{0}^{\infty} x(t) e^{-st} dt \]  

(7)

assuming that the integral can be evaluated at the upper and lower limits to yield well-defined and bounded values. Associated with this transform \( X(s) \), and equivalent to the statement that the transform exists, is the signal’s Region of Convergence (ROC). If the transform exists, the ROC exists, and vice versa. The ROC is that open half of the \( s \)-plane that lies to the right of all singularities (poles) of \( X(s) \). At any point \( s = s_0 \) that lies inside the ROC, the following relationship is true.

\[ X(s_0) = \int_{0}^{\infty} x(t) e^{-s_0 t} dt \]  

(8)
Therefore, at any point \( s = s_0 \) in the Region of Convergence, the form of the defining equation of the Laplace transform is still applicable, with the complex variable \( s \) replaced by the complex number \( s_0 \). The result of evaluating (8) is a numerical value that is a real number if \( s_0 \) is real and a complex number if \( s_0 \) is complex.

In the problem being considered here, the closed-loop system is assumed to be internally stable. All roots of \( \Delta_{CL}(s) \) lie in the open left-half plane. Thus, the Region of Convergence for the transform \( E(s) \) in (6) is that part of the \( s \)-plane to the right of the right-most closed-loop pole. The assumption of internal stability implies that the origin is included in the Region of Convergence for \( E(s) \). Therefore, (8) can be evaluated for \( E(s) \) and \( e(t) \) with \( s_0 = 0 \).

C. Form of the Step Response

Applying (8) to \( E(s) \) and \( e(t) \) with \( s_0 = 0 \) yields the following result

\[
\int_0^\infty e(t)e^{-st}dt = E(s)|_{s=0} = E(0) = 0 \cdot \frac{D(0)}{\Delta_{CL}(0)} = 0
\]

The exponential term inside the integral in (9) is equal to 1. The error signal has an initial value \( e(0) = 1 \) and a final value \( e(\infty) = 0 \), so \( e(t) \) is not identically equal to 0. However, the results stated in (9) indicate that the area under the curve of \( e(t) \) over all time is equal to 0. Therefore, the error signal \( e(t) \) must take on negative values. This means that when the open-loop system has two (or more) poles at the origin of the \( s \)-plane, a step input produces an error signal that is negative for part of the time. Since \( e(t) = r(t) - c(t) \), a negative error signal means that \( c(t) > r(t) \). Thus, the Type 2 system exhibits overshoot in its response to a step input.

D. Concluding Remarks

The result in (9) implies that whenever the open-loop system \( G(s) \) has 2 or more poles at \( s = 0 \), then the closed-loop system will always have overshoot in the step response. Thus, if the system to be controlled, \( G_p(s) \), is Type 2 (or higher), then there is no proper cascade (series) compensator \( G_c(s) \) that provides internal stability for the closed-loop system such that there is no overshoot in the step response.

Note that if the open-loop system were Type 1 rather than Type 2, then the expression for \( E(s) \) in (6) would not have had the term \( s \) in the numerator. In that case, (9) would evaluate to a non-zero number. Since the area under the \( e(t) \) curve would not be restricted to being 0, overshoot in the step response would not be guaranteed. The Type 1 system may or may not have overshoot in the step response, unlike the Type 2 (or higher) system which will always exhibit overshoot.

In order to achieve zero overshoot in the step response and maintain internal stability for a Type 2 system, minor loop compensation around the plant can be used. An example of this is the use of rate feedback, where the derivative of the output signal is fed back to the plant input, with the output signal itself being fed back to form the error signal in the usual manner. An implementation of this occurs when a Proportional + Derivative (PD) controller is used in the Derivative on Output Only (DOO) configuration\(^1\). When this is done, the result is an equivalent plant model that is only Type 1 even though the actual plant model is Type 2 or higher\(^2\).

---

\(^1\) Root Locus Design Example #3, G.O. Beale, available at http://ece.gmu.edu/~gbeale/ece_421/dsgn_421_root_03.pdf