First-Order System Example #1

A. Overview

Two different first-order systems will be presented in this example. The first system, \( G_1(s) \) will have its one open-loop pole located at the origin of the \( s \)-plane, that is, at \( s = 0 \). The second system, \( G_2(s) \) will have its one open-loop pole located at some other place along the real axis. Neither of these systems will have a zero in its transfer function. Thus, neither of these systems represents the most general first-order system model, but the two of them together do represent the most general strictly proper first-order system models. The reason for differentiating between \( G_1(s) \) and \( G_2(s) \) in terms of the location of the open-loop pole will be discussed in the following sections.

For each of these systems, the corresponding closed-loop transfer function will be developed under the assumption of unity feedback, that is, with \( H(s) = 1 \). The closed-loop responses of these systems to a unit step input and to a unit ramp will be developed using partial fraction expansion. Several transient response and steady-state response characteristics will be defined in terms of the parameters in the open-loop transfer functions. These characteristics will be useful in comparing the time-domain performance of different first-order systems, and they will also serve as a basis for the more general characteristics of second-order systems to be studied later.

B. System #1

B.1 The System Models

The first system to be considered is given by the following transfer function which will be placed in the forward path of a unity-feedback closed-loop system.

\[
G_1(s) = \frac{K}{s}, \quad K > 0
\]  

(1)

where \( K \) is a positive real number serving as the gain of the open-loop system. This transfer function can also be written in the following forms by simple algebraic manipulation.

\[
G_1(s) = \frac{1}{\left(\frac{1}{K}\right)s} = \frac{1}{Ts}
\]  

(2)

where \( T \triangleq 1/K \) is defined as the time constant of the system. All of the various time-domain characteristics that will be defined for this first-order system will be expressed in terms of the time constant.

Using the last form for the expression in (2), the closed-loop system under unity feedback for this system is given by

\[
T_{CL-1}(s) = \frac{G_1(s)}{1 + G_1(s)} = \frac{\frac{1}{T}s}{1 + \frac{1}{Ts}} = \frac{1}{s + 1/T} = \frac{1}{s + \frac{1}{T}} = \frac{K}{s + K}
\]  

(3)

The closed-loop pole for this system is located at \( s = -1/T = -K \). Since \( K > 0 \), the closed-loop system is guaranteed to be stable.

B.2 The Unit Step Response

Assume that the reference input to the closed-loop system is the unit step function, which has the Laplace transform \( R(s) = 1/s \). The transform of the output signal is

\[
C_1(s) = T_{CL-1}(s) \cdot R(s) = \frac{1}{s + 1/T} \cdot \frac{1}{s} = \frac{1}{s(s + 1/T)}
\]  

(4)

The output signal in the time domain, \( c_1(t) \), can be found from (4) using partial fraction expansion.

\[
C_1(s) = \frac{1}{s(s + 1/T)} = \frac{A_1}{s} + \frac{A_2}{s + 1/T}
\]  

(5)

\[
A_1 = [s \cdot C_1(s)]_{s=0} = 1, \quad A_2 = [(s + 1/T) \cdot C_1(s)]_{s=-1/T} = -1
\]  

(6)

Therefore, the output signal is

\[
T_{CL-1}(s) \text{ Step Response: } \quad c_1(t) = e^{-t/T} \left[ 1 - \frac{t}{s + 1/T} \right] = 1 - e^{-t/T}, \quad t \geq 0
\]  

(7)
The output has an initial value $c_1(0) = 0$, and the output asymptotically approaches $c_1(t) = 1$ as $t \to \infty$. Since the value of the step input was assumed to be equal to 1 and the final value of the output is also equal to 1, the error between input and output as $t \to \infty$ is equal to 0. The error in the final value of the step response output relative to the input is known as the steady-state error. For the open-loop system defined by (2), the closed-loop step response has zero steady-state error for any value of $T > 0$.

Since the output signal never actually equals its final value for finite values of $t$, there need to be ways of expressing the speed of response of the system that allows different systems to be compared in a meaningful fashion. The first of the transient response characteristics that will be defined is the settling time $T_s$. (Be careful to distinguish between the symbol $T_s$ for settling time and the product of the time constant $T$ and the Laplace variable $s$, namely, $Ts$.) The settling time will be defined in the following way.

**Definition 1**: The settling time $T_s$ is the length of time it takes the system output in response to a step input to reach and stay within a specified tolerance about the final value of the output. In this class, the tolerance for settling time in the step response will always be ±2% of the final value, assuming that the final value is non-zero.

For the output signal given in (7), the final value is $\lim_{t \to \infty} [c(t)] = 1$, so the settling time is determined by the amount of time it takes the output to reach the value $c(t) = 0.98$ and stay within the range $0.98 \leq c(t) \leq 1.02$. Thus, if the time constant of a system is $1$ second, then the settling time as given in (9) is 4 seconds. If a system has a time constant of 3.2 microseconds, then the settling time in response to a step input will be 12.8 microseconds, or if the time constant is 1.6 days, then the settling time will be 6.4 days. The time constant of the open-loop system serves as a scale factor for the time-domain characteristics. Regardless of the value of the time constant $T$, the settling time will be (approximately) 4 times the value of $T$.

The other transient response characteristic that will be defined for the first-order system is rise time $T_r$. The rise time is a measure of how quickly the system responds to the input in terms of getting close to the final value the first time. The usual choice for rise time, at least for first-order systems, is the time it takes the output to increase from 10% of its final value to 90% of its final value. This is known—for obvious reasons—as the 10%-90% rise time. The rise time can be determined from (7) by setting $c(t) = 0.1$ and solving for the resulting $t = t_{10}$ and repeating that with $c(t) = 0.9$, which occurs at $t = t_{90}$. The rise time is $T_r = t_{90} - t_{10}$. The calculations give the following results.

\[
c(t_{10}) = 0.1 = 1 - e^{-t_{10}/T} \Rightarrow t_{10} = -T \ln [0.9] = 0.1054 \cdot T \tag{10}
\]
\[
c(t_{90}) = 0.9 = 1 - e^{-t_{90}/T} \Rightarrow t_{90} = -T \ln [0.1] = 2.3026 \cdot T \tag{11}
\]
\[
T_r = t_{90} - t_{10} = 2.1972T \approx 2.2T \tag{12}
\]

Just as with the settling time, the rise time of the step response is scaled by the system time constant $T$.

Figure 1 graphically shows the definitions of the settling time and rise time in the closed-loop step response of the first-order system. In both graphs in the figure, the independent variable is the dimensionless normalized time $t/T$. As $t \to \infty$, the step input signal gets close to its finite value of $1$. This is known—for obvious reasons—as the steady-state error.
Normalizing the time by the time constant allows the characteristics of the response to be shown without worrying about the actual time scale. This will be particularly useful in the study of second-order systems where there are two design parameters in the transfer function. Using normalized time for the first-order system is equivalent to letting the time constant be $T = 1$ second.

The definition for settling time is shown in the top graph of Fig. 1. Settling time for the first-order system is defined to be the time at which the output reaches 0.98 (actually 0.98168). From (9), the settling time is $T_s = 4T$, so in terms of normalized time, the settling time is $T_s/T = 4$. The definition for rise time is shown in the bottom graph.

Figure 2 shows the step responses of the closed-loop system for four different values of $T$. The independent variable in this figure is actual time $t$ in seconds, so the differences in the speeds of response can be seen. It is clear from the curves plotted in the figures that the smaller the value of $T$, the quicker the response of the system, both in rise time and settling time. This is also obvious from equations (9) and (12).

B.3 The Unit Ramp Response

The second reference input signal that will be applied to the closed-loop system is a unit ramp function whose transform is $R(s) = 1/s^2$. The transform of the output signal is

$$C_1(s) = T_{CL-1}(s) \cdot R(s) = \frac{1}{T} \cdot \frac{1}{s + 1/T} = \frac{1}{T} \cdot \frac{1/s}{s + 1/T}$$

As before, partial fraction expansion can be used to obtain the output signal.

$$C_1(s) = \frac{1}{s^2 (s + 1/T)} = \frac{A_1}{s} + \frac{A_2}{s + 1/T} + \frac{A_3}{s + 1/T}$$

Therefore, the unit ramp response is

$$T_{CL-1}(s) \text{ Ramp Response: } c_1(t) = \left[ \frac{1}{s^2} - \frac{1}{s + 1/T} + \frac{1}{s + 1/T} \right] = t - T + T e^{-t/T}, \quad t \geq 0$$

As $t \to \infty$, the exponential term in (17) goes to 0, so the output signal increases linearly with time—just as the reference input does—with an offset equal to the time constant $T$. The slope of the output signal and the slope of the unit ramp input signal are both equal to 1, so after the transient part of the response decays, the input and output signals are parallel. The steady-state error in the ramp response is the difference between input and output signals after the transient response has decayed. This error is

$$e_{ss} = \lim_{t \to \infty} [r(t) - c(t)] = \lim_{t \to \infty} \left[ t - T + T e^{-t/T} \right] = t - (t - T) = T$$

Thus, the system time constant $T$ not only serves as a scale factor in the step response, it also determines the accuracy in the ramp response for large values of $t$. The smaller the value that $T$ has, the more rapidly the output responds to a step input and the more accurately the output follows a ramp input.

Figures 3 and 4 show the response of the closed-loop system to a unit ramp input. The linearly increasing dashed lines in the figures represent the unit ramp input signal. The steady-state error in the ramp response is defined in Fig. 3, which is plotted in normalized time units. The steady-state error is the vertical distance between the input and output signals after the transient part of the response has decayed to zero. For this system, by $t/T = 13.5$ units, the transient term $e^{-13.5}$ is negligibly small, and the vertical distance between the curves is $e_{ss}/T = 1$ unit.

The ramp responses for four different values of $T$ are illustrated in Fig. 4, with a zoomed view of the curves shown in the bottom graph. By $t = 15$ seconds, the output curves for $T = 0.25$, $T = 0.5$, and $T = 1$ second essentially have reached their final slopes; the vertical distance from the reference input down to each of the curves is the corresponding value of $T$. For $T = 2$ seconds, the error is within 0.01% of its steady-state value at $t = 15$ seconds, so steady-state has been reached for all practical considerations for that system also. For larger values of $T$, it would take correspondingly longer values of time before the ramp response reached steady-state. In general, the settling time for the ramp response is longer than the settling time for the step response for a given value of $T$. 
Fig. 1. Definitions of settling time and rise time in the step response for $T_{CL-1}(s) = 1/(Ts + 1)$. 
Step Response of $1/(Ts + 1)$ for $T = 0.25, 0.5, 1, 2$ seconds

Fig. 2. Step responses for $T_{CL}(s) = 1/(Ts + 1)$.
Fig. 3. Definition of steady-state error in the ramp response of $T_{CL-1}(s) = 1/(Ts + 1)$. 
Zoomed View of the Ramp Responses

Fig. 4. Ramp responses for $T_{CL-1}(s) = 1/(Ts + 1)$. 
C. System #2
C.1 The System Models

The second system model that will be discussed has the following forward-path transfer function.

\[ G_2(s) = \frac{K}{s - p}, \quad K > 0, \quad p \neq 0 \]

where the open-loop pole is located at \( s = p \). If \( p > 0 \), the open-loop pole is in the right-half plane—\( G_2(s) \) is open-loop unstable—and if \( p < 0 \), the open-loop pole is in the left-half plane and \( G_2(s) \) is open-loop stable. The closed-loop system with (19) and unity feedback is

\[ T_{CL-2}(s) = \frac{G_2(s)}{1 + G_2(s)} = \frac{K}{1 + \frac{s}{s-p}} = \frac{K}{s + (K-p)} \]

The closed-loop pole is located at \( s = -(K-p) \). If \( p < 0 \), the closed-loop system is always stable (with \( K > 0 \)). If \( p > 0 \), the closed-loop system is stable if \( K > p \). In the rest of these notes, we will assume that \( p = -1 \), so both the open-loop and closed-loop systems are stable.

Letting \( p = -1 \) for purposes of illustration in these notes, the closed-loop transfer function can be manipulated into a standard form in the following way.

\[ T_{CL-2}(s) = \frac{K}{s + (K+1)} = \frac{K}{(K+1) \left[ \left( \frac{1}{K+1} \right) s + 1 \right]} = \frac{K}{(K+1) \left[ \left( \frac{1}{K+1} \right) s + 1 \right]} \]

\[ T_{CL-2}(s) = \frac{KT}{Ts + 1} = \frac{K}{s + 1/T} \]

where \( T \triangleq 1/(K+1) \) is defined as the system time constant. Note that the last term in (22) has exactly the same form as the system in (3), except now \( 1/T = K+1 \) rather than \( K \). In the general case when the open-loop pole is located as \( s = p \), the expression for the time constant would be \( T = 1/(K-p) \).

C.2 The Unit Step Response

If a unit step function is applied as the reference input signal to the closed-loop system, the output signal can be derived using partial fraction expansion just as before.

\[ C_2(s) = T_{CL-2}(s) \cdot R(s) = \frac{K}{s + (K+1)} \cdot \frac{1}{s + 1/T} = \frac{K}{s(s + 1/T)} \]

\[ C_2(s) = \frac{K}{s(s + 1/T)} = \frac{B_1}{s} + \frac{B_2}{s + 1/T} \]

\[ B_1 = [s \cdot C_2(s)]_{s=0} = KT, \quad B_2 = [(s + 1/T) \cdot C_2(s)]_{s=-1/T} = -KT \]

and the output signal in the time domain is

\[ T_{CL-2}(s) \text{ Step Response:} \quad c_2(t) = \mathcal{L}^{-1} \left[ \frac{KT}{s} - \frac{KT}{s + 1/T} \right] = \left( \frac{K}{K+1} \right) \left( 1 - e^{-t/T} \right), \quad t \geq 0 \]

The output has an initial value \( c_2(0) = 0 \), and the output asymptotically approaches \( c_2(t) = KT = K/(K+1) \) as \( t \to \infty \). Since the value of the step input was assumed to be equal to 1 and the final value of the output is different from 1, the error between input and output as \( t \to \infty \) does not go to 0; there is a non-zero steady-state error in the step response for this system. Since \( K > 0 \), \( K/(K+1) < 1 \) and the final value of the output is less than the value of the reference input. The larger the value of \( K \), the closer the final value of the output will be to 1, but it will always be less than 1. (If the open-loop pole is in the right-half plane, \( p > 0 \), but the closed-loop system is stable, \( K > p \), then the final value of the output signal will be greater than 1.)

The transient response of the output signal is still controlled by the time constant \( T \), just as with system \( T_{CL-1}(s) \); there is just a slightly different definition for \( T \) in this second case. The definitions for settling time and rise time are exactly the same is given previously in (9) and (12). The only difference is that with \( c_2(t) \) the final value is \( K/(K+1) \) rather than 1 as it was for \( c_1(t) \). Settling time is still the time required for the output to reach \( 0.98K/(K+1) \), and the rise time is the time required for the output to go from 10% to 90% of \( K/(K+1) \).
Figure 5 shows the step responses for the output signal in (26) for gain values of $K = 0.5, 1, 2, 5, 10$. As the plots indicate, the output signal gets closer to the reference step input value of 1 as the value of $K$ increases. The corresponding values for the time constant are $T = 1/(K + 1) = 0.6667, 0.5, 0.3333, 0.1667, 0.0909$. The final values for the output are $K/(K + 1) = 0.3333, 0.5, 0.6667, 0.8333, 0.9091$. If the open-loop pole was at some location other than $p = -1$, the values of $T$ and the final values of $c_2(t)$ would change.

C.3 The Unit Ramp Response

For a unit ramp reference input signal, the transform of the output would be

$$C_2(s) = T_{CL-2}(s) \cdot R(s) = \frac{K}{s+1/T} \cdot \frac{1}{s^2} - \frac{K}{s^2(s+1/T)}$$

which can be expanded in the partial fraction format.

$$C_2(s) = \frac{K}{s^2(s+1/T)} = \frac{B_1}{s^2} + \frac{B_2}{s} + \frac{B_3}{s+1/T}$$

$$B_1 = [s^2 \cdot C_2(s)]_{s=0} = KT,$$
$$B_3 = [(s+1/T) \cdot C_2(s)]_{s=-1/T} = KT^2$$

$$B_2 = \left. \frac{d}{ds} [s^2 \cdot C_2(s)] \right|_{s=0} = \left. \frac{d}{ds} \left( \frac{K}{s+1/T} \right) \right|_{s=0} = \left[ \frac{(s+1/T)(0) - K(1)}{(s+1/T)^2} \right]_{s=0} = -KT^2$$

The ramp response is
Ramp Response of $K T / (T s + 1)$ for $K = 0.5, 1, 2, 5, 10$

Fig. 6. Ramp response plots for $T_{CL-2}(s) = K T / (T s + 1)$ with $T = 1 / (K + 1)$ and $K = 0.5, 1, 2, 5, 10$.

$T_{CL-2}(s)$ Ramp Response: 
\[ c_2(t) = \mathcal{L}^{-1} \left[ \frac{K T}{s^2} - \frac{K T^2}{s} + \frac{K T^2}{s + 1/T} \right] = K T \left[ t - T + T e^{-t/T} \right], \quad t \geq 0 \quad (31) \]

The ramp response in (31) is seen to be identical to the response in (17) for $T_{CL-1}(s)$ except for the scale factor $K T = K / (K + 1)$. The same scale factor appeared in the step response for $T_{CL-2}(s)$, having the effect of producing a non-zero steady-state error. The scale factor also affects the steady-state error in the ramp response. The slope of the output signal is no longer equal to 1, it is equal to $K / (K + 1)$. Thus, the reference input signal and output signal will diverge with time. The steady-state error in this case will be

\[ e_{ss} = \lim_{t \to \infty} [r(t) - c(t)] = t - \lim_{t \to \infty} \left[ 1 - \left( \frac{K}{K+1} \right) \right] T \left[ 1 + \left( \frac{K}{K+1} \right) \right] T = \infty \quad (33) \]

No matter what the value of $K$ is, the ramp response will have infinitely large steady-state error.

Figure 6 shows the ramp responses for this system using the same values of $K$ as for the step responses. Clearly, the output signal is diverging from the reference input in each case. The output signal will stay ‘close’ to the reference input for longer periods of time for larger $K$, but the vertical distance between input and output will always become infinitely large for this system model.
D. Summary

This example has looked at the time-domain properties of two standard first-order system models. The definitions of settling time, rise time, and steady-state error have been presented, and their relationships to the system models have been derived. The concept of a system time constant $T$ has been introduced, and it was shown that all the characteristics in the step and ramp responses depend on the value of $T$, except for the steady-state error in the ramp response of system model #2, which always has infinite steady-state error.

The difference between the open-loop transfer functions for system #1 and system #2 is the location of the open-loop pole. In the first system it is constrained to be at $s = 0$, while in the second system it could be located at any point on the real axis as long as the closed-loop system was stable. When the open-loop pole is at the origin, the step response always has zero steady-state error, and the ramp response has a non-zero but finite steady-state error. When the open-loop pole is not located at the origin, the step response has a non-zero finite steady-state error and the ramp response has an infinitely large steady-state error. This trend will be seen to also exist in higher-order system models and different types of input signals as well.

Although first-order systems cannot display all possible behaviors in the time domain, they do accurately model a number of real physical systems. The voltage across the capacitor in a series RC circuit is one example. The temperature in a room being heated or cooled would also be modeled as a first-order system. The linearized model of certain ceramic materials being heated by microwave energy is also first-order.