Stability of Dynamical Systems

Introduction

Classical Control

Stability of a system is of paramount importance. In general, an unstable system is both useless and dangerous. When a system is unstable, state and/or output variables are becoming unbounded in magnitude over time—at least theoretically. In practice, at some point in time, electronic amplifiers will saturate and mechanical components will reach their physical limits of motion. In any event, the system is no longer operating in a well-behaved manner.

In courses in classical control theory, the systems being considered are generally linear and time-invariant, and stability is generally analyzed in terms of the locations of the poles of a transfer function, that is, the zeros of the denominator polynomial. In order to be stable, the poles of the transfer function for a continuous-time system must all lie strictly in the left half of the complex s-plane. The poles must all have strictly negative real parts. For a discrete-time system to be stable, the poles of the transfer function must lie in the interior of a circle of unit radius centered at the origin of the complex z-plane. The magnitude of the poles must all be less than one.

The specific type of stability that is described by these requirements on pole locations is known as Bounded-Input, Bounded-Output (BIBO) stability. This and other types of stability will be defined in a later section. For a system to be BIBO stable, any input signal \( u(t) \) applied to the system that is bounded in magnitude \( \|u(t)\| < \infty \) must produce an output that also remains bounded for all time.

To illustrate the effects of pole locations on stability, consider the following three transfer functions. Only \( H_1(s) \) is BIBO stable; all the poles are in the left half of the s-plane. \( H_2(s) \) has poles on the \( j\omega \) axis, and \( H_3(s) \) has a pole in the right half of the s-plane. Therefore, neither of them is BIBO stable. Figure 1 shows the responses of these three systems to a sinusoidal input signal \( u(t) = \sin(\omega t) \) whose frequency is \( \omega = \sqrt{11} \) rad/sec, the same as the imaginary values of the complex poles in \( H_2(s) \).

\[
H_1(s) = \frac{30}{(s + 4)(s + 2 + j2)(s + 2 - j2)} = \frac{Y_1(s)}{U(s)} \tag{1}
\]

\[
H_2(s) = \frac{60}{(s + 6)(s^2 + 11)} = \frac{Y_2(s)}{U(s)}, \quad H_3(s) = \frac{5}{(s - 1)(s + 2)(s + 3)} = \frac{Y_3(s)}{U(s)} \tag{2}
\]

Assuming that the major characteristics of the output signals are shown in Fig. 1, it is clear from the plots that the outputs from transfer functions \( H_2(s) \) and \( H_3(s) \) are growing without bound, while the output from \( H_1(s) \) is remaining within a finite bound. The output \( y_2(t) \) is growing linearly with time due to the \( t\sin(\omega t) \) term, and \( y_3(t) \) is growing exponentially due to the \( e^t \) term.

Discrete-time transfer functions with stability properties similar to those for the continuous-time systems in Eqns. (1) and (2) are shown below.

\[
H_1(z) = \frac{3}{(z + 0.4)(z - 0.2 + j0.2)(z - 0.2 - j0.2)} = \frac{Y_1(z)}{U(z)} \tag{3}
\]

\[
H_2(z) = \frac{6}{(z - 0.6)(z^2 + 1)} = \frac{Y_2(z)}{U(z)}, \quad H_3(z) = \frac{5}{(z - 2)(z + 0.2)(z - 0.3)} = \frac{Y_3(z)}{U(z)} \tag{4}
\]
Figure 1: Responses of three systems to a sinusoidal input.
State Space Control

As discussed above, stability in the classical sense working with transfer functions is a function of the locations of the poles of the system transfer function. In state space models for linear time-invariant systems, stability is determined by the locations of the eigenvalues of the $A$ matrix in $x(t) = Ax(t) + Bu(t)$ or $x(k+1) = Ax(k) + Bu(k)$. For the transfer functions of continuous-time systems in (1) and (2), possible $A$ matrices are

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -32 & -24 & -8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -66 & -11 & -6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{bmatrix} \tag{5}$$

The eigenvalues for these matrices are $\lambda_1 = \{-4, -2 \pm j2\}$, $\lambda_2 = \{-6, \pm j\sqrt{11}\}$, $\lambda_3 = \{1, -2, -3\}$. These are the same values as the poles of the three transfer functions. This will always be the case when there are no pole-zero cancellations in the transfer functions. For the transfer functions of discrete-time systems in Eqns. (3) and (4), the corresponding $A$ matrices could be

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.032 & 0.08 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.6 & -1.2 & 1.6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.12 & -0.14 & 2.1 \end{bmatrix} \tag{6}$$

The eigenvalues for these matrices are $\lambda_1 = \{-0.4, 0.2 \pm j0.2\}$, $\lambda_2 = \{0.6, 0.5 \pm j0.866\}$, $\lambda_3 = \{2, -0.2, 0.3\}$. As before, these eigenvalues are the same as the poles of the respective transfer functions.

Equilibrium Points

Concepts

Stability analysis for linear time-invariant (LTI) systems is fairly simple. A system that is linear and time-invariant is either stable or unstable. There are some different definitions of stability—defined in the next section—but stability for this type of system does not depend on time or on the present location of the state vector in the $n$-dimensional state space. Stability analysis for time-varying linear systems and for nonlinear systems is more complicated.

An equilibrium point $x_e$ in state space $\Sigma$ is a point at which a system will remain in the absence of external inputs or disturbances. Therefore, for a continuous-time system described by the general state equation $\dot{x}(t) = f [x(t), u(t), t]$, the point $x_e$ is an equilibrium point if

$$\dot{x_e}(t) = 0 = f [x_e(t), 0, t] \tag{7}$$

If a system is at an equilibrium point at time $t = t_0$, and no external forces act on the system, it will remain at the equilibrium point for all $t \geq t_0$. For linear time-invariant systems, (7) becomes

$$\dot{x_e}(t) = 0 = Ax_e \tag{8}$$

An equilibrium point $x_e$ for a discrete-time system is defined in a similar manner. Since the system state remains at an equilibrium point when there are no external inputs applied, it follows that $x(k+1) = x(k)$ when $x(k) = x_e$. For the general nonlinear discrete-time system, the state equation $x(k+1) = f [x(k), u(k), k]$ at an equilibrium point is

$$x_e = f (x_e, 0, k) \tag{9}$$

and for a linear time-invariant, discrete-time system, the state equation is $x_e = Ax_e$. 

3
The analysis of system stability is closely connected with the concept of equilibrium points. As illustrated in (7)–(9), a system whose state is at an equilibrium point will remain there unless an external input acts on the system. The question about stability then becomes what happens to the system if a momentary external input—intentionally applied or not—does perturb the system away from the equilibrium point. There are three possible answers to this question:

1. The system state returns to the equilibrium point.
2. The system state does not return to the equilibrium point but does remain “close” to that point.
3. The system state diverges from the equilibrium point.

Examples of Equilibrium Points

Example 1 Consider an unforced, discrete-time linear system described by $x(k + 1) = Ax(k)$. The eigenvalues of the $A$ matrix are $\lambda = \{1, 0.6, -0.5\}$, and the $A$ matrix in companion form is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & 0.2 & 1.1 \end{bmatrix} \quad (10)$$

For a point in $\Sigma$ to be an equilibrium point, $x_e = Ax_e$ or $(I_n - A)x_e = 0$. The matrix $(I_n - A)$ and its Row-Reduced Echelon (RRE) form are

$$I_3 - A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0.3 & -0.2 & -0.1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (11)$$

The first row of the RRE form of the matrix shows that $x_1 = x_3$, the second row shows that $x_2 = x_3$, and the third row shows that one element of $x_e$ is arbitrary, which in this case must be $x_3$. Therefore, this system has an infinite number of equilibrium points given by $x_e = [\alpha \alpha \alpha]^T$, with $\alpha$ being arbitrary. The origin of state space is one of the equilibrium points, with $\alpha = 0$. The reason for the infinite number of equilibrium points is the fact that $A$ has an eigenvalue at $z = 1$, which represents pure integration in discrete time.

Example 2 Now consider an unforced, continuous-time linear system described by $\dot{x}(t) = Ax(t)$. The eigenvalues of the $A$ matrix are $\lambda = \{0, -2, -3\}$. The $A$ matrix in companion form and its RRE equivalent (since $Ax_e = 0$) are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (12)$$

The first row of the RRE matrix shows that $x_2 = 0$, the second row shows that $x_3 = 0$, and the last row shows that there is one arbitrary variable, which obviously for this system must be $x_1$. Therefore, for this system, there is an infinite number of equilibrium points which are given by $x_e = [\alpha \ 0 \ 0]^T$. The pole at $s = 0$ represents a pure integrator in continuous time, which is the reason for the infinite number of $x_e$ values.
Example 3

For this last example, consider a linear continuous-time system without any integrators. The eigenvalues are $\lambda = \{-1, -2, -4\}$. The $A$ matrix in companion form and its RRE equivalent are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad (13)$$

The first row of the RRE matrix shows that $x_1 = 0$, the second row shows that $x_2 = 0$, and the last row shows that $x_3 = 0$. Therefore, the only equilibrium point for this system is the origin of state space, $x_e = 0$.

These three examples have shown that for linear systems, if the system $A$ matrix contains an eigenvalue that corresponds to pure integration ($\lambda = 1$ in discrete time or $\lambda = 0$ in continuous time), then there are an infinite number of equilibrium points. However if there are no such eigenvalues in the $A$ matrix, then there is a unique equilibrium point which is the origin, that is, the only equilibrium point is $x_e = 0$.

**Stability Definitions**

Several definitions of stability are presented here, following the material in the course textbook\(^1\).

The first two definitions are for unforced systems, that is, $u(t) = 0$ for all $t$.

**Definition 4** An equilibrium point $x_e$ is **stable** if for any given value of a parameter $\epsilon > 0$ there exists a number $\delta(\epsilon)$ such that if $\|x(t_0) - x_e\| < \delta$, then the state vector satisfies $\|x(t) - x_e\| < \epsilon$ for all $t > t_0$. The relationship $\delta \leq \epsilon$ is required. This type of stability is also known as **stable in the sense of Lyapunov** (i.s.L.). Therefore, a system that is stable in the sense of Lyapunov remains close to the equilibrium point following a perturbation.

For a linear time-invariant system to be stable in the sense of this definition, all eigenvalues of the $A$ matrix must be in the region of stability (open left-half of the $s$-plane for continuous-time systems or interior of the unit circle in the $z$-plane for discrete-time systems) except for the possibility of un-repeated eigenvalues on the boundary of stability ($j\omega$ axis for continuous-time systems or $|z| = 1$ for discrete-time systems).

**Definition 5** An equilibrium point $x_e$ is **asymptotically stable** if it (a) is stable as in the previous definition and (b) in addition there exists a number $\delta' > 0$ such that if $\|x(t_0) - x_e\| < \delta'$, the state vector satisfies $\lim_{t \to \infty} \|x(t) - x_e\| = 0$. Thus, for asymptotic stability, the state returns to the equilibrium point.

For a linear time-invariant system to be asymptotically stable, all eigenvalues of the $A$ matrix must lie strictly in the region of stability (open left-half of the $s$-plane or interior of the unit circle in the $z$-plane). In that case, $x_e = 0$ is the only equilibrium point. If a linear time-invariant system is asymptotically stable, then it is also globally asymptotically stable since any initial state would converge to the origin if the system had no external input.

**Example 6** Consider three unforced continuous-time LTI systems with the same initial condition $x(0) = [1 \ 0 \ 1]^T$. The system matrices are

Each of the definitions for stability can occur. Each of the definitions assume that the input signal \( u(t) \) or \( u(k) \) is bounded in norm for all time. Some examples of bounded inputs include step functions, sinusoidal signals, decaying exponentials, and combinations of these. Examples of unbounded inputs include ramp or parabolic functions and growing exponentials.

**Definition 7** An input is said to be bounded if there exists a constant \( K > 0 \) such that \( \|u\| \leq K < \infty \) for all time. A system is said to be bounded-input, bounded-state (BIBS) stable if there exists a constant \( \delta > 0 \), which may depend on \( K \) and on \( x(0) \), such that \( \|x\| \leq \delta \) for any bounded input and any initial condition.

**Definition 8** Let the input \( u \) be bounded, with \( K_m \) being the least upper bound (the smallest number such that \( \|u\| \leq K_m \)). Then a system is said to be bounded-input, bounded-output (BIBO) stable if there exists a constant \( \alpha > 0 \) such that \( \|y\| \leq \alpha K_m \) for all time. This is the type of stability most often considered in a classical controls course.

### Stability Analysis for Linear Systems

#### Time-Varying Systems

Here it is assumed that the systems are described by linear, time-varying, continuous-time state and output equations in the standard form

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)
\]  

(15)

When \( u(t) = 0 \), the solution to (15) is

\[
x(t) = \Phi(t, t_0) x(t_0)
\]  

(16)

where \( \Phi(t, t_0) \) is the state transition matrix. Since the input signal \( u(t) \) is zero, the types of stability that can be investigated are stable in the sense of Lyapunov (stable i.s.L.) and asymptotic stability. For the system to be stable i.s.L., the state must remain in some neighborhood of an equilibrium point after a perturbation. If the \( A \) matrix does not have an eigenvalue \( \lambda = 0 \), then \( x_e = 0 \) is the only equilibrium point. If \( A \) does have an eigenvalue \( \lambda = 0 \), then there are an infinite number of equilibrium points. To cover both possibilities with the minimum of notation, define the perturbation of the state away from any equilibrium point by \( \Delta x(t) = x(t) - x_e \).

The norm of the distance of the state \( x(t) \) from the equilibrium point \( x_e \) can be used to determine the constraints on the transition matrix in order for the system to be stable i.s.L.

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-4 & -4 & -1
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -2 & -3
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-5 & -7 & -3
\end{bmatrix}
\]  

(14)

The eigenvalues are \( \lambda_1 = \{-1, j2, -j2\} \), \( \lambda_2 = \{-1, -2, 0\} \), and \( \lambda_3 = \{-1, -1 + j2, -1 - j2\} \). From the stability definitions, the first two systems are stable but not asymptotically stable, while the third system is asymptotically stable. Figure 2 shows the state responses for each of these systems. It is clear from the plots that for the first two systems all the states stay near the origin but not all the states converge to the origin. The states in the first system continue to oscillate due to the complex conjugate poles. Two of the states in the second system do converge to the origin, but the other state \( x_1 \) converges to a constant non-zero value due to the eigenvalue \( \lambda = 0 \). For the third system all the states converge to the origin.
Figure 2: Examples of stability in the sense of Lyapunov and asymptotic stability.
\[ \| \Delta_x(t) \| = \| \Phi(t, t_0) \cdot \Delta_x(t_0) \| \leq \| \Phi(t, t_0) \| \cdot \| \Delta_x(t_0) \| \] (17)

where the last form of the expression in (17) comes directly from the definition of the norm for a square matrix, which is \( \| W \| = \text{sup} \{ \| Wx \| / \| x \| \} \) over all \( x \) such that \( \| x \| \neq 0 \).

Based on Definition 4, if the norm of the transition matrix is bounded, such that \( \| \Phi(t, t_0) \| \leq N \) for all \( t \geq t_0 \), then the system will be stable in the sense of Lyapunov for any value of \( \epsilon > 0 \) by choosing the perturbation at the initial time to satisfy \( \delta(\epsilon) \leq \epsilon/N \). This is a necessary and sufficient condition for the system to be stable i.s.L.

For a system to be asymptotically stable, the state response must both stay within a neighborhood of the equilibrium point over time and also converge to that point at \( t \to \infty \). The constraints on the transition matrix for asymptotic stability are

\[ \| \Phi(t, t_0) \| \leq N \text{ for all } t \geq t_0 \quad \text{and} \quad \| \Phi(t, t_0) \| \to 0 \text{ as } t \to \infty \] (18)

If the input \( u(t) \) is non-zero, then Bounded-Input, Bounded-State and Bounded-Input, Bounded-Output stability may be studied. The first requirement for both BIBS and BIBO stability is that the system be stable i.s.L. The solution to Eqn. (15) when \( u(t) \neq 0 \) in terms of a deviation from the equilibrium point is

\[ \Delta_x(t) = \Phi(t, t_0) \cdot \Delta_x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) \, d\tau \] (19)

When the norm is taken of both sides of the previous equation, and the triangle property of norms is used (twice), constraints for BIBS stability are obtained.

\[ \| \Delta_x(t) \| = \left\| \Phi(t, t_0) \cdot \Delta_x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) \, d\tau \right\| \] (20)

\[ \| \Delta_x(t) \| \leq \| \Phi(t, t_0) \cdot \Delta_x(t_0) \| + \left\| \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) \, d\tau \right\| \] (21)

\[ \| \Delta_x(t) \| \leq \| \Phi(t, t_0) \| \cdot \| \Delta_x(t_0) \| + \int_{t_0}^{t} \| \Phi(t, \tau) B(\tau) u(\tau) \| \, d\tau \] (22)

From (22), the constraints for BIBS stability that must be satisfied for \( x(t) \) to remain bounded for all initial conditions and all bounded input signals are

\[ \| \Phi(t, t_0) \| \leq N \text{ for all } t \geq t_0 \quad \text{and} \quad \int_{t_0}^{t} \| \Phi(t, \tau) B(\tau) \| \, d\tau \leq N_1 \text{ for all } t \geq t_0 \] (23)

To determine the constraints for BIBO stability we will look at the solution to the output equation \( y(t) = C(t)x(t) + D(t)u(t) \). From (19) the output signal is

\[ y(t) = C(t) \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} C(\tau) \Phi(t, \tau) B(\tau) u(\tau) \, d\tau + D(t)u(t) \] (24)

The previous equation can be simplified in two ways. First, we may assume that the initial condition \( x(t_0) \) came about from some input having been applied to the system in the time interval from \( t = -\infty \) to \( t = t_0 \). That eliminates the first term and changes the lower limit on the integration from \( t_0 \) to \( -\infty \). The second step is to use the sifting property of the impulse function in reverse to take the term \( D(t)u(t) \) inside the integral. The result of these two steps gives us the following.
\[
y(t) = \int_{-\infty}^{t} [C(\tau) \Phi(t, \tau) B(\tau) + \delta(t - \tau) D(\tau)] u(\tau) d\tau = \int_{-\infty}^{t} W(t, \tau) u(\tau) d\tau
\]  

The \( m \times r \) matrix \( W(t, \tau) = C(\tau) \Phi(t, \tau) B(\tau) + \delta(t - \tau) D(\tau) \) is known as the weighting matrix. It is the impulse response matrix for the system. The \( (i,j) \) element of \( W(t, \tau) \) is the response at time \( t \) at the \( i^{th} \) output terminal due to an impulse function applied at time \( \tau \) at the \( j^{th} \) input terminal, the inputs at the other terminals being identically zero.\(^2\) Since the input is assumed to be bounded, \( \|u(t)\| \leq K_m \) for all \( t \), the requirement that must be satisfied by the system for BIBO stability is that there exists a constant \( M > 0 \) such that

\[
\int_{-\infty}^{t} \|W(t, \tau)\| d\tau \leq M \quad \text{for all } t
\]

It should be obvious from this discussion that a matrix norm plays a large part in the conditions for each of the types of stability that have been discussed. Various definitions of matrix norms exist. One common and convenient definition for a matrix norm is the largest singular value of the matrix. The matrix does not have to be square for this. For any matrix \( S \), the largest singular value \( \sigma \) is the square root of the maximum eigenvalue of \( S^T S \), that is

\[
\sigma = \sqrt{\max \{\lambda(S^T S)\}}
\]  

For example, if the \( 4 \times 6 \) matrix \( S \) is given by

\[
S = \begin{bmatrix}
9 & 0 & 10 & 0 & 20 & 2.8 \\
0 & -0.3 & -9 & 0 & 0 & -3.13 \\
0 & 1 & -0.3 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}
\]

the eigenvalues of \( S^T S \) are \( \{607.8615, 72.0994, 1.8304, 1.0256, 0, 0\} \). The maximum singular value of \( S \) is \( \sigma = \|S\| = \sqrt{607.8615} = 24.6548 \).

**Time-Invariant Systems**

Stability of linear time-invariant (LTI) systems has already been discussed, at least from the BIBO perspective. System stability depends entirely on the locations of the eigenvalues of the \( A \) matrix. The state transition matrix for continuous-time systems is \( \Phi(t, \tau) = e^{A(t-\tau)} \), and for discrete-time systems, \( \Phi(k, 0) = A^k \). Assume that \( A \) has \( n \) linearly independent eigenvectors (no generalized eigenvectors). Then \( A = M^{-1}AM \) is the diagonal matrix of eigenvalues \( \lambda_i \) of \( A \), with \( M = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \) being the modal matrix whose columns are the eigenvectors of \( A \). The transition matrices can be written as

\[
e^{A(t-\tau)} = Me^{A(t-\tau)}M^{-1}, \quad A^k = MA^kM^{-1}
\]

With a change of basis defined by \( x = Mq \), and letting \( t_0 = 0 \) for convenience, the solutions to the state equations for the homogeneous case are

\[
x(t) = e^{At}x(0) \quad \rightarrow \quad x(t) = Me^{At}M^{-1}x(0) \quad \rightarrow \quad q(t) = e^{At}q(0) \tag{30}
\]

\[
x(k) = A^kx(0) \quad \rightarrow \quad x(k) = MA^kM^{-1}x(0) \quad \rightarrow \quad q(k) = \Lambda^kq(0) \tag{31}
\]

Therefore, \( q(t) = e^{\lambda_t}q_i(0) \) for the continuous-time system, and \( q_i(k) = \lambda_i^k q_i(0) \) in discrete time. Letting \( \lambda_i = \sigma_i + j\omega_i \), the continuous-time solution can be written as \( q_i(t) = e^{\sigma_i t}e^{j\omega_i t}q_i(0) \). From

these expressions it can be seen that system stability is determined by the real parts of the eigenvalues $\sigma_i$ in continuous time, and by the magnitudes of the eigenvalues $|\lambda_i|$ in discrete time since $e^{\sigma_i t}$ and $\lambda_i^k$ must be bounded for all $t$ and $k$, respectively.

Although the above discussion on LTI stability was for a system with a full set of linearly independent eigenvectors—systems having either all distinct eigenvalues or having full degeneracy for any repeated eigenvalues—the same conclusions may be drawn concerning stability when there are generalized eigenvectors as well. Stability requirements for linear, time-invariant systems may be summarized as follows.

1. Continuous-Time Systems

   (a) Unstable: If any eigenvalue $\lambda_i$ has its real part $\sigma_i > 0$, the system is unstable since there will be a growing exponential in the response.

   (b) Stable in the sense of Lyapunov: All repeated eigenvalues must have their real parts strictly negative, $\sigma_i < 0$, since the response associated with a repeated eigenvalue will be multiplied by $t$. Distinct eigenvalues may have their real parts either negative or zero, $\sigma_i \leq 0$. The terms associated with distinct eigenvalues with $\sigma = 0$ will not decay to zero, but they will not grow without bound.

   (c) Asymptotic stability: For a system to be asymptotically stable, all eigenvalues must have strictly negative real parts, $\sigma_i < 0$, since each term must have a decaying exponential in it. A system that is asymptotically stable is also BIBS and BIBO stable.

2. Discrete-Time Systems

   (a) Unstable: If any eigenvalue $\lambda_i$ lies outside the unit circle centered at the origin of the complex plane, $|\lambda_i| > 1$, the system is unstable since there will be a growing term in the response.

   (b) Stable in the sense of Lyapunov: All repeated eigenvalues must lie strictly inside the unit circle, $|\lambda_i| < 1$, since the response associated with a repeated eigenvalue will be multiplied by $k$. Distinct eigenvalues may lie inside or on the boundary of the unit circle, $|\lambda_i| \leq 1$. The terms associated with distinct eigenvalues with $|\lambda_i| = 1$ will not decay to zero, but they will not grow without bound.

   (c) Asymptotic stability: For a system to be asymptotically stable, all eigenvalues must lie strictly inside the unit circle, $|\lambda_i| < 1$, since each term must decay to zero with increasing time. A system that is asymptotically stable is also BIBS and BIBO stable.

The matrices $A_1$, $A_2$, and $A_3$ shown in Eqn. (5) for continuous-time systems are asymptotically stable ($A_1$), stable i.s.L. ($A_2$), and unstable ($A_3$). The discrete-time matrices in Eqn. (6) are also asymptotically stable ($A_1$), stable i.s.L. ($A_2$), and unstable ($A_3$). This is easily verified by inspection of the eigenvalues.

**Eigenvalues and Stability for Time-Varying Systems**

Since stability for linear time-invariant systems is totally a function of the locations of the eigenvalues of the $A$ matrix, we might ask if we can use the same results for linear time-varying (LTV) systems. The answer is sometimes yes and sometimes no, but we always have to be cautious in doing so.

Time-varying systems can come about in many ways. Nonlinear system models are often linearized about a nominal trajectory. As the linear model is evaluated at different points in state
space, the coefficients in the matrices change. Controller coefficients for aircraft are intentionally
varied depending on flight conditions, and controller coefficients for ships are often a function of
speed. Each of these scenarios causes the elements in the linear system matrices to vary over time,
even if they are not explicit functions of time.

If the elements in the matrices do not vary “too rapidly” and if the eigenvalues are all well
within the region of stability at each set of values, then in many cases the system will remain
stable. However, there is no guarantee of that in general. Even when the eigenvalues themselves
are stable and constant for a time-varying system, there is no guarantee of stability, as the following
example shows.

The model used here is the one in Example 10.5 on page 359 of the text. It illustrates the fact
that the eigenvalues of the system matrix in an LTV model do not necessarily define stability. The
state space model with \( u(t) = 0 \) is

\[
\dot{x}(t) = \begin{bmatrix}
-1 + \alpha \cos^2(t) & 1 - \alpha \sin(t) \cos(t) \\
-1 - \alpha \sin(t) \cos(t) & -1 + \alpha \sin^2(t)
\end{bmatrix} x(t) = A(t)x(t)
\]

(32)

When the eigenvalues of \( A(t) \) are computed through \(|\lambda - A(t)| = 0\), all the terms involving \( \sin(t) \)
and \( \cos(t) \) cancel out, leaving the following characteristic polynomial.

\[
\Delta(\lambda) = 0 = \lambda^2 + (2 - \alpha) \lambda + (2 - \alpha)
\]

(33)

The eigenvalues depend on the parameter \( \alpha \) but do not depend at all on time \( t \). Therefore, for a
fixed value of \( \alpha \), the eigenvalues of \( A(t) \) are constants. They are the solutions to

\[
\lambda = \frac{- (2 - \alpha)}{2} \pm \frac{\sqrt{(2 - \alpha)^2 - 4 (2 - \alpha)}}{2} = \frac{(\alpha - 2)}{2} \pm \frac{\sqrt{\alpha - 2 (\alpha + 2)}}{2}
\]

(34)

The top graph in Fig. 3 shows the eigenvalue locations for \(-4 \leq \alpha \leq 6\). This is the root locus
plot of \(-\alpha (\lambda + 1)/ (\lambda^2 + 2\lambda + 2)\). If the parameter \( \alpha \) is any real value greater than 2, then
the eigenvalues are both real, with one being positive and the other negative. For linear time-invariant
systems, this fact would mean that the system is unstable. However, if the parameter is less than
2, then the eigenvalues have strictly negative real parts, being complex conjugates for \(-2 < \alpha < 2\)
and real for \( \alpha \leq -2 \). For LTI systems, this fact would guarantee asymptotic stability. To see what
it means for this time-varying system, the state transition matrix needs to be investigated.

The solution for the state vector for the system defined in Eqn. (32) is \( x(t) = \Phi(t, 0) x(0) \), with
the transition matrix being

\[
\Phi(t, 0) = \begin{bmatrix}
e^{(\alpha-1)t} \cos(t) & e^{-t} \sin(t) \\
-e^{(\alpha-1)t} \sin(t) & e^{-t} \cos(t)
\end{bmatrix}
\]

(35)

It is clear from the elements in the first column of \( \Phi(t, 0) \) that for \( \alpha > 1 \) the transition matrix
contains exponential terms that grow without bound with increasing time. Therefore, the system
is unstable for \( \alpha > 1 \) even though all the eigenvalues of \( A(t) \) have strictly negative real parts for
all \( \alpha < 2 \). The bottom left graph in Fig. 3 shows the response of the state vector to the initial
condition \( x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix} \) when \( \alpha = 1.5 \). It is clear from this graph that the system is unstable
with that value for \( \alpha \). When the parameter value is changed to \( \alpha = 0.5 \), the state response to the
same initial condition is shown in the bottom right graph, which is seen to be an asymptotically
stable response.

Thus, this system is stable i.s.L. for \( \alpha = 1 \), unstable for \( \alpha > 1 \), and asymptotically stable
for \( \alpha < 1 \). This example illustrates the fact that for linear time-varying systems, the location
of eigenvalues may provide some useful information, but in general more complete information is
needed in order to make a final decision on stability.
Figure 3: Eigenvalues of the time-varying $A(t)$ matrix and state responses for $\alpha = 1.5$ and $\alpha = 0.5$. 