Full-State Feedback Design for a Multi-Input System

A. Introduction

The open-loop system is described by the following state space model.

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
\]

\[\begin{bmatrix}
-4 & 8 & -1.5 \\
0 & 0 & 1 \\
-8 & 14 & -3 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0.5 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & -0.5 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

(1)

(2)

The model has three states and represents a multi-input, multi-output (MIMO) system. The number of states is \( n = 3 \) and the number of inputs and outputs is \( r = m = 2 \). The eigenvalues of the open-loop system are \( \lambda_{OL} = \{-8.71, 0.856 \pm j0.43\} \). Since the open-loop system is unstable, some form of feedback is required to produce a stable system. In this example, full-state feedback will be used. The controllability matrix for the open-loop system is

\[
P = \begin{bmatrix}
B & AB & A^2B
\end{bmatrix} = \begin{bmatrix}
1 & 0.5 & -4 & -3.5 & 28 & 32 \\
0 & 0 & 0 & 1 & -8 & -7 \\
0 & 1 & -8 & 7 & 56 & 63
\end{bmatrix}, \quad \text{Rank}(P) = r_P = 3 = n
\]

(3)

Since the rank of \( P \) is equal to \( n \), the system is completely controllable. Therefore, the closed-loop eigenvalues can be placed at arbitrary places in the complex plane as long as one of the eigenvalues is complex, its complex conjugate is also an eigenvalue. For this example, the closed-loop eigenvalues will be placed at \( \lambda_{CL} = \{-2, -3, -4\} \). Although this choice of closed-loop eigenvalues may not give satisfactory performance, they are asymptotically stable and will serve for this example.

Three solutions methods will presented in this example:

1) Row-Reduced Echelon (RRE)
2) Singular Value Decomposition (SVD)
3) The \textit{place} function in MATLAB.

A feedback gain matrix will be computed with each of these methods, and the results will be compared in terms of verifying that the closed-loop eigenvalues are all the same, the directions of the eigenvectors, and the time-domain response to an initial condition. No attempt will be made with the RRE and SVD methods to optimize the performance relative to any specific performance index. The primary purpose of the example is to see how the design methods work with a multi-input system.

B. Design Overview

With full-state feedback, \( u(t) = -Kx(t) + v(t) \), and the closed-loop state equations are

\[
\dot{x}(t) = (A - BK)x(t) + Bv(t)
\]

(4)

The closed-loop eigenvalues and eigenvectors are related by

\[
[A - BK] \psi_i = \lambda_i \psi_i \quad \Rightarrow \quad [\lambda_i I - A + BK] \psi_i = 0
\]

(5)

Since the \( r \times n \) feedback gain matrix \( K \) is unknown, Eqn. (5) can be rewritten as shown below which will lead to a procedure for computing the value for \( K \).

\[
\begin{bmatrix}
\lambda_i I - A & BK
\end{bmatrix} \begin{bmatrix}
\psi_i \\
\psi_i
\end{bmatrix} = 0 \Rightarrow \begin{bmatrix}
\lambda_i I - A & B
\end{bmatrix} \begin{bmatrix}
\psi_i \\
K \psi_i
\end{bmatrix} = 0 \Rightarrow \begin{bmatrix}
\lambda_i I - A & B
\end{bmatrix} \xi_i = 0
\]

(6)

where \( \psi_i \) is the \( n \)-dimensional eigenvector associated with eigenvalue \( \lambda_i \), and \( \xi_i \) is a vector of dimension \( n + r \) (5 in this example).

For each desired closed-loop eigenvalue \( \lambda_i \), the matrix \( S_i = \begin{bmatrix}
\lambda_i I - A & B
\end{bmatrix} \) will be formed, and the Row-Reduced Echelon (RRE) and Singular Value Decomposition (SVD) techniques will be used to find the vector \( \xi_i \). Once that is done for each \( \lambda_i \), the gain matrix \( K \) can be computed by partitioning the \( \xi_i \) into \( \psi_i \) and \( K \psi_i \). The procedure for each method is described in this example.
C. Row-Reduced Echelon

For each eigenvalue, the matrix \( S_i = \begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \) is formed and the RRE method applied to transform the matrix into the row-reduced equivalent. The procedure is the same as illustrated in previous examples and will not be shown in detail here. Only the starting and ending points will be shown. The MATLAB function `rref` is used to implement the RRE method.

For \( \lambda = -2 \), the matrices are:

\[
S_1 = \begin{bmatrix} -2I - A & B \end{bmatrix} = \begin{bmatrix} 2 & -8 & 1.5 & 1 & 0.5 \\ 0 & -2 & -1 & 0 & 0 \\ 8 & -14 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -0.2857 & 0.0536 \\ 0 & 1 & -0.1429 & -0.0357 \\ 0 & 0 & 0.2857 & 0.0714 \end{bmatrix} \xi_1 = 0 \tag{7}
\]

The elements of the vector \( \xi_1 \) will be denoted by \( \xi_{11}, \xi_{12}, \ldots, \xi_{15} \). The three rows of the RRE matrix establish the following relationships between the elements of the vector.

\[
\xi_{11} = 0.2857\xi_{14} - 0.0536\xi_{15}, \quad \xi_{12} = 0.1429\xi_{14} + 0.0357\xi_{15}, \quad \xi_{13} = -0.2857\xi_{14} - 0.0714\xi_{15} \tag{8}
\]

The fourth and fifth elements of \( \xi_1 \), which correspond to \( K\psi_1 \), are arbitrary. They will be set to 1 and -1, respectively, so

\[
\xi_1 = \begin{bmatrix} 0.3393 & 0.1071 & -0.2143 & 1 & -1 \end{bmatrix}^T.
\]

For \( \lambda = -3 \), the same steps will be followed.

\[
S_2 = \begin{bmatrix} -3I - A & B \end{bmatrix} = \begin{bmatrix} 1 & -8 & 1.5 & 1 & 0.5 \\ 0 & -3 & -1 & 0 & 0 \\ 8 & -14 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -0.1628 & 0.0640 \\ 0 & 1 & -0.0930 & -0.0349 \\ 0 & 0 & 0.2791 & 0.1047 \end{bmatrix} \xi_2 = 0 \tag{9}
\]

The RRE matrix establishes the following relationships between the elements of the vector \( \xi_2 \).

\[
\xi_{21} = 0.1628\xi_{24} - 0.0640\xi_{25}, \quad \xi_{22} = 0.0930\xi_{24} + 0.0349\xi_{25}, \quad \xi_{23} = -0.2791\xi_{24} - 0.1047\xi_{25} \tag{10}
\]

The elements \( \xi_{24} \) and \( \xi_{25} \) are arbitrary and will be set to 1 and 0, respectively. Therefore, the vector for this eigenvalue is

\[
\xi_2 = \begin{bmatrix} 0.1628 & 0.0930 & -0.2791 & 1 & 0 \end{bmatrix}^T.
\]

Finally, for \( \lambda = -4 \), the same procedure is used once again.

\[
S_3 = \begin{bmatrix} -4I - A & B \end{bmatrix} = \begin{bmatrix} 0 & -8 & 1.5 & 1 & 0.5 \\ 0 & -4 & -1 & 0 & 0 \\ 8 & -14 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -0.0893 & 0.0804 \\ 0 & 1 & -0.0714 & -0.0357 \\ 0 & 0 & 0.2857 & 0.1429 \end{bmatrix} \xi_3 = 0 \tag{11}
\]

The RRE matrix establishes the following relationships between the elements of the vector \( \xi_3 \).

\[
\xi_{31} = 0.0893\xi_{34} - 0.0804\xi_{35}, \quad \xi_{32} = 0.0714\xi_{34} + 0.0357\xi_{35}, \quad \xi_{33} = -0.2857\xi_{34} - 0.1429\xi_{35} \tag{12}
\]

As before, the last two elements in the vector are arbitrary. They will be set to 1 and -1, respectively. Therefore, the vector for the eigenvalue \( \lambda = -4 \) is \( \xi_3 = \begin{bmatrix} 0.1696 & 0.0357 & -0.1429 & 1 & -1 \end{bmatrix}^T \).

The columns of the modal matrix \( M \) of eigenvectors are the first \( n = 3 \) rows of the vectors \( \xi_1, \xi_2, \) and \( \xi_3 \). The last two elements of those three vectors form the matrix \( Q \) (remember, this is not the observability matrix). For this example, \( M \) and \( Q \) are

\[
M_{rre} = \begin{bmatrix} 0.3393 & 0.1628 & 0.1696 \\ 0.1071 & 0.0930 & 0.0357 \\ -0.2143 & -0.2791 & -0.1429 \end{bmatrix}, \quad Q_{rre} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \tag{13}
\]

Since \( Q = \begin{bmatrix} K\psi_1 & K\psi_2 & K\psi_3 \end{bmatrix} \) and \( M = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix} \), it should be clear that \( Q = KM \), and the gain matrix can be solved from

\[
K_{rre} = Q_{rre}M_{rre}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0.3393 & 0.1628 & 0.1696 \\ 0.1071 & 0.0930 & 0.0357 \\ -0.2143 & -0.2791 & -0.1429 \end{bmatrix}^{-1} = \begin{bmatrix} 3.5849 & -15.0094 & -6.4953 \\ -8.4528 & 22.7170 & 2.6415 \end{bmatrix} \tag{14}
\]

The closed-loop matrix \( A_{CL} = A - BK_{rre} \) and its eigenvalues are given by

\[
A_{CL} = \begin{bmatrix} -3.3585 & 11.6509 & 3.6745 \\ 0 & 0 & 1 \\ 0.4528 & -8.7170 & -5.6415 \end{bmatrix}, \quad \lambda_{CL} = \{-2, -3, -4\} \tag{15}
\]
The eigenvalues are seen to lie at their specified locations, so the gain matrix was computed correctly. The final closed-loop system using the RRE method is

\[
\begin{bmatrix}
-3.3585 & 11.6509 & 3.6745 \\
0 & 0 & 1 \\
0.4528 & -8.7170 & -5.6415 \\
\end{bmatrix}
\begin{bmatrix}
x(t) \\
v(t) \\
x(t) \\
\end{bmatrix}
+
\begin{bmatrix}
1 & 0.5 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
y(t) \\
x(t) \\
\end{bmatrix}
= 0.
\]

(16)

A second gain matrix based on the RRE method will be computed and the resulting eigenvalues compared with those already obtained. For this matrix, the values for the arbitrary elements in the \( \xi \) vectors will be \( \xi_{14} = \xi_{24} = \xi_{34} = 0 \) and \( \xi_{15} = \xi_{25} = \xi_{35} = 1 \). The \( Q \), \( M \), and \( K \) matrices using these values are

\[
M_{rre2} = \begin{bmatrix}
-0.0536 & -0.0640 & -0.0804 \\
0.0357 & 0.0349 & 0.0357 \\
-0.0714 & -0.1047 & -0.1429 \\
\end{bmatrix},
\quad Q_{rre2} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 1 \\
\end{bmatrix},
\quad K_{rre2} = \begin{bmatrix}
0 & 0 & 0 \\
16 & 40 & -6 \\
\end{bmatrix}
\]

(17)

The closed-loop matrix \( A_{CL} = A - BK_{rre2} \), its eigenvalues, and the final closed-loop system are given by

\[
A_{CL} = \begin{bmatrix}
-12 & -12 & 1.5 \\
0 & 0 & 1 \\
-24 & -26 & 3 \\
\end{bmatrix},
\quad \lambda_{CL} = \{-2, -3, -4\}
\]

(18)

\[
\begin{bmatrix}
-12 & -12 & 1.5 \\
0 & 0 & 1 \\
-24 & -26 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x(t) \\
v(t) \\
x(t) \\
\end{bmatrix}
+
\begin{bmatrix}
1 & 0.5 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
y(t) \\
x(t) \\
\end{bmatrix}
= 0.
\]

(19)

The second gain matrix using the RRE method also has properly located the closed-loop eigenvalues. With this gain matrix, it can be seen that the control signal \( u_1(t) = 0 \) since the first row of \( K_{rre2} \) is all zeros. Even with that, the closed-loop system is completely controllable from the external input \( v(t) \). A comparison of the time-domain responses of these two designs is given in the last section of the example.

D. **Singular Value Decomposition**

The matrix \( S_1 = [\lambda_i I - A \quad B] \) is formed for each eigenvalue, just as with the RRE method. Once that is done, singular value decomposition (SVD) is performed on each \( S_i \). The MATLAB function \( \text{svd} \) can be used for this. The syntax for this function is \( [u, sig, V] = \text{svd}(S) \). For our example, \( S \) has \( n = 3 \) rows and \( n + r = 5 \) columns. The matrix \( V \) returned by the \( \text{svd} \) function has \( n + r \) rows and columns. The right-most \( r = 2 \) columns of \( V \) form an orthonormal basis for the null space of \( S \), and the eigenvectors \( \psi_i \) and the terms \( K\psi_i \) are formed from linear combinations of those two columns from the \( V_i \) matrix corresponding to eigenvalue \( \lambda_i \). Once those linear combinations are chosen, the \( Q \) and \( M \) matrices can be formed and the gain matrix \( K \) computed.

The matrices \( S_1, S_2, \) and \( S_3 \) are given in Eqns. (7), (9), and (11), respectively. They are the same for both RRE and SVD, so they will not be repeated. The last two columns of \( V_1, V_2, \) and \( V_3 \) from the \( \text{svd} \) function are

\[
V_1 = \begin{bmatrix}
0.2619 & -0.0590 \\
0.1317 & 0.0327 \\
-0.2634 & -0.0654 \\
0.9190 & -0.0198 \\
0.0122 & 0.9954 \\
\end{bmatrix},
\quad V_2 = \begin{bmatrix}
0.1538 & -0.0677 \\
0.0884 & 0.0322 \\
-0.2652 & -0.0965 \\
0.9477 & -0.0263 \\
0.0070 & 0.9922 \\
\end{bmatrix},
\quad V_3 = \begin{bmatrix}
0.0850 & -0.0826 \\
0.0684 & 0.0326 \\
-0.2736 & -0.1305 \\
0.9556 & -0.0366 \\
0.0042 & 0.9868 \\
\end{bmatrix}
\]

(20)

The columns of each of the \( V_i \) will be denoted \( V_{11} \) and \( V_{12} \). The linear combinations that were chosen for this example are

\[
\begin{bmatrix}
\psi_1 \\
K\psi_1 \\
\end{bmatrix} = V_{11} + V_{12},
\quad \begin{bmatrix}
\psi_2 \\
K\psi_2 \\
\end{bmatrix} = V_{21} - V_{22},
\quad \begin{bmatrix}
\psi_3 \\
K\psi_3 \\
\end{bmatrix} = -V_{31} + V_{32}
\]

(21)

and the resulting matrices are

\[
M_{\text{svd}} = \begin{bmatrix}
0.2029 & 0.2216 & -0.1676 \\
0.1644 & 0.0562 & -0.0358 \\
-0.3289 & -0.1687 & 0.1431 \\
\end{bmatrix},
\quad Q_{\text{svd}} = \begin{bmatrix}
0.8992 & 0.9740 & -0.9923 \\
1.0075 & -0.9852 & 0.9826 \\
\end{bmatrix}
\]

(22)

The feedback gain matrix is

\[
K_{\text{svd}} = Q_{\text{svd}}M_{\text{svd}}^{-1} = \begin{bmatrix}
0.7337 & -15.1696 & -9.8662 \\
-6.9811 & 24.2563 & 4.7569 \\
\end{bmatrix}
\]

(23)

The closed-loop matrix \( A_{CL} = A - BK_{\text{svd}} \) and its eigenvalues are given by
\[
A_{CL} = \begin{bmatrix}
-1.2431 & 11.0414 & 5.9877 \\
0 & 0 & 1 \\
-1.0189 & -10.2563 & -7.7569
\end{bmatrix}, \quad \lambda_{CL} = \{-2, -3, -4\} \tag{24}
\]

The final closed-loop system using the SVD method is
\[
\dot{x}(t) = \begin{bmatrix}
-1.2431 & 11.0414 & 5.9877 \\
0 & 0 & 1 \\
-1.0189 & -10.2563 & -7.7569
\end{bmatrix} x(t) + \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v(t), \quad y(t) = \begin{bmatrix} 1 & 0 & -0.5 \end{bmatrix} x(t) \tag{25}
\]

**E. MATLAB Place Function**

Use of the *place* function in MATLAB returns the gain matrix in a single step. The input arguments to the function are the A and B matrices and the desired closed-loop eigenvalues. For this example, the gain matrix using the *place* function is
\[
K_{place} = \text{place}(A, B, \lambda_{CL}) = \begin{bmatrix}
2.9720 & -1.2364 & -2.3342 \\
-7.9015 & 22.0014 & 2.9788
\end{bmatrix} \tag{26}
\]

The closed-loop matrix \(A_{CL} = A - BK_{place}\) and its eigenvalues are given by
\[
A_{CL} = \begin{bmatrix}
-3.0212 & -1.7643 & 0.6552 \\
0 & 0 & 1 \\
-0.0985 & -8.0014 & -5.9788
\end{bmatrix}, \quad \lambda_{CL} = \{-2, -3, -4\} \tag{27}
\]

The final closed-loop system using the *place* function is
\[
\dot{x}(t) = \begin{bmatrix}
-3.0212 & -1.7643 & 0.6552 \\
0 & 0 & 1 \\
-0.0985 & -8.0014 & -5.9788
\end{bmatrix} x(t) + \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v(t), \quad y(t) = \begin{bmatrix} 1 & 0 & -0.5 \end{bmatrix} x(t) \tag{28}
\]

**F. Comparison of Designs**

From examination of Eqns. (14), (17), (23), and (26), it is clear that the four design have produced four different feedback gain matrices. In spite of this, the closed-loop eigenvalues are at the specified locations for each of the designs. Whenever the number of inputs \(r > 1\), there are an infinite number of gain matrices that will produce the same eigenvalues of \(A - BK\). However, the eigenvectors and time-domain performances may be quite different for different designs. Thus, the elements of the \(\xi\) vectors that have been considered arbitrary, are arbitrary only in the sense of producing a gain matrix to place the closed-loop eigenvalues at specified locations. However, they are not arbitrary when it comes to guaranteeing good performance as well as eigenvalue locations.

The time-domain responses for the first RRE design, the SVD design, and the *place* function design are shown in Fig. 1. The comparison of the two RRE designs is shown in Fig. 2. The closed-loop systems given in Eqns. (16), (19), (25), and (28) were simulated using the *lsim* function in MATLAB. The initial condition for the state vector in each case was \(x(0) = [0 \ 1 \ 0]^T\), and the external reference input signal \(v(t) = 0\).

In order to compare the responses in a quantitative manner, a quadratic performance index (PI) was calculated for each system. The PI is
\[
J = \sum (x^T Q_{pi} x + u^T R_{pi} u) \tag{29}
\]

The quadratic terms in (29) were computed at each time instant in the simulation arrays, and the summation was over the number of time steps. The weighting matrices \(Q_{pi}\) and \(R_{pi}\) were used to normalize the variables based on the largest peak values of the designs for each of the state and control variables. Those matrices were
\[
Q_{pi} = \begin{bmatrix}
(1/1.5)^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & (1/2.4)^2
\end{bmatrix}, \quad R_{pi} = \begin{bmatrix}
(1/16)^2 & 0 & 0 \\
0 & (1/40)^2
\end{bmatrix} \tag{30}
\]

The following table shows the values of the performance index for the various designs. The total values computed from (29) are shown, as well as the values for only the state variables and the values for only the control variables.

<table>
<thead>
<tr>
<th></th>
<th>RRE</th>
<th>RRE-2</th>
<th>SVD</th>
<th>Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>252.2</td>
<td>199.3</td>
<td>200.7</td>
<td>143.2</td>
</tr>
<tr>
<td>(x) only</td>
<td>231.3</td>
<td>156.4</td>
<td>179.7</td>
<td>118.2</td>
</tr>
<tr>
<td>(u) only</td>
<td>20.84</td>
<td>42.87</td>
<td>20.97</td>
<td>25.01</td>
</tr>
</tbody>
</table>
It is clear from the table that in terms of the quadratic PI, the place function in MATLAB has produced the best results. That function will generally give very good results, but it should be remembered that nothing was done in the RRE or SVD designs to try and optimize the performance. The free (arbitrary) variables in those designs were chosen rather arbitrarily.

A second measure of performance to consider is the angles between the eigenvectors. The most linearly independent that two eigenvectors can be is when they are orthogonal, that is, having an angle of 90° between them. The closer the angles are to 0° or 180°, the closer the vectors are to being collinear. Also, the less linearly independent the eigenvectors are, the closer the modal matrix is to being singular. The following table shows the angles between the eigenvectors for each of the designs and the value of the determinant of the modal matrix $M$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>RRE</th>
<th>RRE-2</th>
<th>SVD</th>
<th>Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{12}$</td>
<td>26.5°</td>
<td>7.85°</td>
<td>23.2°</td>
<td>83.6°</td>
</tr>
<tr>
<td>$\theta_{13}$</td>
<td>9.60°</td>
<td>11.9°</td>
<td>158°</td>
<td>168°</td>
</tr>
<tr>
<td>$\theta_{23}$</td>
<td>20.4°</td>
<td>4.11°</td>
<td>176°</td>
<td>97.0°</td>
</tr>
<tr>
<td>$</td>
<td>M</td>
<td>$</td>
<td>$-1.57 \cdot 10^{-3}$</td>
<td>$3.71 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

For the place function, eigenvectors 1 and 2 are nearly orthogonal, as are eigenvectors 2 and 3. The magnitude of the determinant for that design is larger than that of the other designs by two or more orders of magnitude. The eigenvectors for the second RRE design are nearly collinear, and that design has the smallest magnitude for $|M|$.

Although all the gains produced the same closed-loop eigenvalues, there are clearly differences between them in terms of performance. It appears that the gain matrix from the place function is overall the best design. In terms of the quadratic performance index $J$, the second RRE design and the SVD design would be next best, and the first RRE design would be worst. However, when the linear independence of the eigenvectors is checked, the second RRE design is the worst. If the place function is not available, the RRE method or SVD method can be used to determine an appropriate gain matrix. However, care is needed in choosing the free variables in those methods. One goal in determining those free variables might be to make the eigenvectors as close to orthogonal as can be achieved for the problem at hand.
Fig. 1. State and control responses for the RRE, SVD, and place function designs with an initial condition of $[0 \ 1 \ 0]^T$. 
Fig. 2. State and control responses for the two RRE designs with an initial condition of $[0 \ 1 \ 0]^T$. 