Optimal Polynomial Control for Discrete-Time Systems

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# Contents

I  Introduction 3

II  The Division Algorithm 5
   A. Overview ............................................................................. 5
   B. Systematic Approach ....................................................... 7

III  Developing the Optimal Control 9
   A. Polynomial Performance Index ............................................ 9
   B. The Optimal Control Law ................................................ 11

IV  The Closed-Loop System 13

V  Example Problem 14
   A. 1-Step Control ................................................................. 15
   B. Model Following Control ................................................ 15
   C. Modified 1-Step Control .................................................. 17

VI  Summary and Concluding Remarks 20

References 21

## List of Figures

1  Structure for discrete-time optimal control with polynomial plant model and performance index. ................................................................. 12
2  Response of the system using 1-step control. .................................................. 16
3  Response of the system using model-following control. ............................... 18
4  Response of the system using modified 1-step control. .............................. 19
I. Introduction

In modern and optimal control, a system is often represented by a state space model, which for a deterministic, discrete-time, linear, time-invariant system is a set of $n$ first-order difference equations written in the form

$$x_{k+1} = Ax_k + Bu_k$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$ are the state vector, control input vector, system matrix, and input matrix, respectively. This provides a system description in terms of the internal state variables, which may or may not have any physical significance. If a model for the measured output is necessary, it would have the form

$$y_k = Cx_k + Du_k$$

where $y \in \mathbb{R}^r$, $C \in \mathbb{R}^{r \times n}$, and $D \in \mathbb{R}^{r \times m}$ are the output vector, output matrix, and direct feedthrough matrix, respectively.

In this presentation, we will consider discrete-time linear systems described in terms of an input-output representation. The model is in the form of an $n^{th}$-degree difference equation, which for a single-input, single-output (SISO) system becomes

$$y(k+n) + a_1y(k+n-1) + \cdots + a_{n-1}y(k+1) + a_n y(k) = b_0u(k+m) + b_1u(k+m-1) + \cdots + b_{m-1}u(k+1) + b_mu(k)$$

where $m \leq n$, $y(k)$ is the output signal, $u(k)$ is the input signal, $a_i$ and $b_i$ are scalars, and $k$ represents an arbitrary point in discrete time. As seen in (3), $y(k+n)$ can be computed in terms of previous values of the output and input. Thus, this equation is recursive and has the form of an infinite impulse response (IIR) digital filter [2], [3] or a deterministic autoregressive moving average (DARMA) filter [4].

To provide a shorthand notation, we will introduce the forward shift operator

$$q^n y(k) = y(k+n)$$

and backward shift operator

$$q^{-n} y(k) = y(k-n)$$

With this notation, (3) can be written as

$$A(q)y(k) = B(q)u(k)$$
where $A(q)$ and $B(q)$ are

$$
A(q) = q^n + a_1 q^{n-1} + \cdots + a_{n-1} q + a_n
$$

(7)

$$
B(q) = b_0 q^m + b_1 q^{m-1} + \cdots + b_{m-1} q + b_m
$$

However, the more generally used way of expressing (3) for designing control systems is in terms of the backward shift operator, giving

$$
A(q^{-1}) y(k) = B(q^{-1}) u(k)
$$

(8)

where $A(q^{-1})$ and $B(q^{-1})$ are defined by

$$
A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{n-1} q^{-(n-1)} + a_n q^{-n}
$$

(9)

$$
B(q^{-1}) = b_0 q^{-(n-m)} + b_1 q^{-(n-m-1)} + \cdots + b_{m-1} q^{-(n-1)} + b_m q^{-n}
$$

(10)

By examining equations (8) – (10), it can be seen that there is a dead time or pure delay time between application of the input signal $u(k)$ and a response at the output signal $y(k)$. The most current input that can affect the output $y(k)$ is $u[k-(n-m)]$. Thus, there is a delay of $(n-m)$ time units between the application of the input and the appearance of any effect of that input at the output signal. We define the dead time or control delay as

$$
d = n - m
$$

(11)

If the input-output relationship is given in the normal transfer function form,

$$
\frac{Y(z)}{U(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}
$$

(12)

d is the relative degree of the transfer function, that is, the difference between the degrees of the denominator and numerator polynomials. Note that the transfer function is expressed in terms of positive powers of $z$. Regardless of where the mathematical model arose, $d$ is the dead time between the control input and the output. In some cases, the real value of $n$ or $m$ will not be apparent until the model is put into transfer function form; for example, open-loop poles at the origin result in some $a_i$ coefficients in (9) that are 0.

Using the definitions of (9), (10), and (11), three additional ways of expressing the input-output relationship of the system with the backward shift operator are:
\[ A(q^{-1})y(k + d) = B'(q^{-1})u(k) \]  
(13)

\[ A(q^{-1})y(k) = B'(q^{-1})u(k - d) \]  
(14)

\[ A(q^{-1})y(k) = q^{-d}B'(q^{-1})u(k) \]  
(15)

where the new input polynomial \( B'(q^{-1}) \) is defined in terms of (10) as:

\[ B'(q^{-1}) = b_0 + b_1q^{-1} + \ldots + b_{m-1}q^{-(m-1)} + b_mq^{-m} = q^d B(q^{-1}) \]  
(16)

The control delay \( d \) is shown explicitly in (13), (14), (15) as an argument in the output variable, as an argument in the input variable, and factored out as a separate term, respectively.

As always, the optimal control problem is to determine an expression for \( u(k) \) that minimizes whatever performance index (PI) that has been chosen, while also satisfying the system equation and any other constraints that have been imposed. Different performance indices will be described in a later section.

Note that when the variable \( u(k) \) appears by itself, it represents the value of the control variable at the single point in time \( k \). However, when \( u(k) \) appears with a polynomial, for example as in the right side of (13), a weighted sum of control input values at different points in time is indicated. The same is true for \( y(k) \) or any other variable appearing in an equation. Thus, in the following example equation

\[ u(k) = M(q^{-1})u(k) + N(q^{-1})y(k) \]  
(17)

where \( M(q^{-1}) \) and \( N(q^{-1}) \) are polynomials in the backward shift operator with \( m_0 = 0 \) and \( n_0 \neq 0 \), the control input at time \( k \) is being expressed in terms of past values of the control input and the present and past values of the output.

II. The Division Algorithm

A. Overview

If we look at (13) and solve for the control input variable \( u(k) \), we get a difference equation that involves past values of the control as well as past, present, and future values of the output signal.

\[ u(k) = \frac{1}{b_0} [-b_1u(k - 1) - b_2u(k - 2) - \ldots - b_mu(k - m) + y(k + d) + a_1y(k + d - 1) + \ldots + a_ny(k + d - n)] \]  
(18)

Therefore, in trying to determine the value of the control input at time \( k \), we need to know future values of the output. Since this would represent a non-causal controller, we replace these future values with predictions. In a regulator design, the predicted future values would generally be the known, constant desired values for the output. In a tracking problem, the prediction would probably be the known reference
signal for the output. In either case, the desired future value of the output at time \( (k + i) \) will be denoted as \( y^*(k + i) \). Thus, in (18), future values of the output \( y(k + i) \), \( i > 0 \), would be replaced by the appropriate value \( y^*(k + i) \). The present value of \( y(k) \) is measurable, and past values have already been measured, so those terms do not change in (18).

In the completely deterministic situation considered in this presentation, (18) could be solved for \( u(k) \) as long as the \( y^*(k + i) \) were available at each time step for all \( i \in [1, d] \). However, the stochastic version of (18) would also have future values of the disturbance process in it, and accurately predicting those values might be difficult. The most common approach to determine the optimal control for either deterministic or stochastic systems is to rewrite (13) so that only one prediction of the output is needed, rather than \( d \) predictions. Using this approach, the system model can be written as

\[
 y(k + d) = \alpha (q^{-1}) y(k) + \beta (q^{-1}) u(k) \tag{19}
\]

where \( \alpha (q^{-1}) \) and \( \beta (q^{-1}) \) are given by

\[
 \alpha (q^{-1}) = \alpha_0 + \alpha_1 q^{-1} + \alpha_2 q^{-2} + \cdots \tag{20}
\]
\[
 \beta (q^{-1}) = \beta_0 + \beta_1 q^{-1} + \beta_2 q^{-2} + \cdots \tag{21}
\]

Since the right side of (19) contains only present and past values of the input and output, \( u(k) \) can be computed, with the only prediction being \( y^*(k + d) \) for the value of \( y(k + d) \). Conversely, note that the right side of (19) is a prediction of what \( y(k + d) \) should be, with the prediction being based on information available at time \( k \). The brute force way to get the form of (19) from (13) is through repeated substitution of time-shifted versions of (13) back into (13) and collecting terms. Thus, expressions of the form

\[
 A (q^{-1}) y(k + d - i) = B' (q^{-1}) u(k - i) \tag{22}
\]

for \( i \in [1, (d - 1)] \) would be substituted into (13) Once terms in \( u(k + i) \) and \( y(k + i) \) were collected, the result would be in the form of (19). A simple example will illustrate the brute force method.

**Example 1:**

\[
 y(k + 2) + a_1 y(k + 1) + a_2 y(k) + a_3 y(k - 1) = b_0 u(k) + b_1 u(k - 1) + b_2 u(k - 2) \tag{23}
\]

The polynomials \( A (q^{-1}) \) and \( B(q^{-1}) \) are seen to be

\[
 A (q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} \tag{24}
\]
\[
 B' (q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2}
\]
Therefore, \( m = 2 \), \( d = 2 \), and apparently \( n = 3 \). Note that in this example, with the system model being given by the difference equation (23), the control delay does not appear to be equal to the difference between \( n \) and \( m \). However, if the Z-transform of (23) were taken and the transfer function formed for this system, it would turn out that there is an additional coefficient \( a_4 \), with a value of 0, corresponding to an open-loop pole at the origin. This could be recognized from the fact that the terms involving \( u(k - i) \) in (23) go further back in time than the terms involving \( y(k - i) \). Thus, the "real" value of \( n \) is \( n = 4 \) and \( d = n - m \) as usual. The open-loop transfer function is

\[
\frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z (z^3 + a_1 z^2 + a_2 z + a_3)} \tag{25}
\]

Now in order to get the system model into the predictor format of (19), Eq. (23) is shifted 1 unit backward in time to produce

\[
y(k + 1) + a_1 y(k) + a_2 y(k - 1) + a_3 y(k - 2) = b_0 u(k - 1) + b_1 u(k - 2) + b_2 u(k - 3) \tag{26}
\]

Equation (26) is solved for \( y(k + 1) \), and the resulting expression is substituted into (23) to yield

\[
y(k + 2) + (a_2 - a_1^2) y(k) + (a_3 - a_1 a_2) y(k - 1) - (a_1 a_3) y(k - 2) = b_0 u(k) + (b_1 - a_1 b_0) u(k - 1) + (b_2 - a_1 b_1) u(k - 2) - (a_1 b_1) u(k - 3) \tag{27}
\]

Comparing (27) and (19), the polynomials are identified as

\[
\alpha (q^{-1}) = -(a_2 - a_1^2) - (a_3 - a_1 a_2) q^{-1} + (a_1 a_3) q^{-2} \tag{28}
\]

\[
\beta (q^{-1}) = b_0 + (b_1 - a_1 b_0) q^{-1} + (b_2 - a_1 b_1) q^{-2} - (a_1 b_1) q^{-3} \tag{29}
\]

Note that in the formulation of (27), there is 1 more coefficient than in the original form of (23). This will be the case when \( d > 1 \). Also note that the leading coefficient of the \( \beta (q^{-1}) \) polynomial is the same as the leading coefficient of the \( B' (q^{-1}) \) polynomial; this will always be the case since \( A (q^{-1}) \) has been defined as a monic polynomial. If the control delay were larger than 2, the above procedure of shifting the original equation, solving for the leading term, and substituting the result into the original equation would be repeated until the final equation was in the form of (19).

\[\blacksquare\]

**B. Systematic Approach**

The non-brute-force method to solve the above problem is to make use of the division algorithm from algebra, which says that when one polynomial is divided by another, the result is a quotient polynomial and a remainder polynomial [5], [6]. Thus,

\[
\frac{a(s)}{b(s)} = q(s) + \frac{r(s)}{b(s)} \tag{30}
\]
where \(a(s)\) and \(b(s)\) are arbitrary polynomials in the indeterminate \(s\), \(q(s)\) is the quotient, and \(r(s)\) is the remainder. To develop this method for the control design application, consider (13), rewriting it in the form

\[
y(k + d) = \frac{1}{A(q^{-1})} \cdot B'(q^{-1}) u(k)
\]  
(31)

and defining

\[
\frac{1}{A(q^{-1})} = F(q^{-1}) + q^{-d} \frac{G(q^{-1})}{A(q^{-1})}
\]  
(32)

The polynomial \(F(q^{-1})\) is the quotient, and \(q^{-d}G(q^{-1})\) is the remainder. Using synthetic division, the degree of \(F(q^{-1})\) could be set at any value just by carrying the division out as far as desired. For our application, the degree of \(F(q^{-1})\) will always be \((d - 1)\). The \(q^{-d}\) term in (32) indicates that the remainder starts with the \(d^{th}\) term in the division. Therefore,

\[
F(q^{-1}) = 1 + f_1 q^{-1} + \cdots + f_{d-1} q^{-(d-1)}
\]  
(33)

\[
G(q^{-1}) = g_0 + g_1 q^{-1} + \cdots + g_{n-1} q^{-(n-1)}
\]  
(34)

Recursive formulas are available to compute these two polynomials, so the process can be automated. These formulas are:

\[
f_i = -\sum_{j=0}^{i-1} f_j a_{i-j} \quad i \in [1, (d-1)] , \quad f_0 = 1
\]  
(35)

\[
g_i = -\sum_{j=0}^{d-1} f_j a_{i+d-j} \quad i \in [0, (n-1)]
\]  
(36)

Coefficients that are needed in the above recursion formulas but that are undefined are assumed to be equal to 0.

Substituting (32) into (31) yields

\[
y(k + d) = F(q^{-1}) B'(q^{-1}) u(k) + \left[ \frac{G(q^{-1}) B'(q^{-1})}{A(q^{-1})} \right] u(k - d)
\]  
(37)

If (31) is shifted backwards in time by \(d\) steps, the resulting expression can be solved for \(u(k - d)\). When that result is substituted into (37), the system model in predictor format becomes

\[
y(k + d) = F(q^{-1}) B'(q^{-1}) u(k) + G(q^{-1}) y(k)
\]  
(38)

The polynomials from (19) can be identified as
\[
\alpha(q^{-1}) = G(q^{-1}) \tag{39}
\]
\[
\beta(q^{-1}) = F(q^{-1}) B'(q^{-1}) \tag{40}
\]

From these definitions, it is clear that the degree of \(\beta(q^{-1})\) is \((d - 1 + m)\) and the degree of \(\alpha(q^{-1})\) is \((n - 1)\).

If there are no parameter uncertainties in the \(A(q^{-1})\) and \(B'(q^{-1})\) polynomials and no stochastic effects in the system model, then \(y(k + d)\) can be computed exactly from (13) once a value for \(u(k)\) has been computed. Conversely, assume that there is a known, desired value for \(y(k + d)\), namely, \(y^*(k + d)\). Then in order to force the actual output \(y(k + d)\) to equal the desired value \(y^*(k + d)\), we can solve (38) for the required \(u(k)\).

\[
u(k) = (1/b_0) \{ y^*(k + d) - [\alpha_0 y(k) + \alpha_1 y(k - 1) + \cdots + \alpha_{n-1} y(k - n + 1) \\
+ \beta_1 u(k - 1) + \beta_2 u(k - 2) + \cdots + \beta_{d+m-1} u(k - d - m + 1)] \} \tag{41}
\]

The procedure to apply this control algorithm is at each time step to:

1. Measure the output signal \(y(k)\);
2. Access the value for the desired output \(y^*(k + d)\);
3. Compute the control signal \(u(k)\) from (41);
4. Apply that \(u(k)\) to control the system until the next time step;
5. Shift the input and output variables backwards one position to be ready for the next set of computations.

This control algorithm is known as a \(1\)-step control since it forces \(y(k)\) to be equal to a reference value at each time step. In the next section we will see that this control is optimal with respect to a particular performance index.

### III. Developing the Optimal Control

#### A. Polynomial Performance Index

A performance index that follows the polynomial format of the system model and that can be tailored for particular applications is given by the following expression.

\[
J = [P(q^{-1}) y(k + d) - Q(q^{-1}) w(k)]^2 + [R(q^{-1}) u(k)]^2 \tag{42}
\]

where

\[
P(q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2} + \cdots + p_n q^{-n_1} \tag{43}
\]
is the weighting polynomial for the system output \( y(k + d) \),

\[
Q(q^{-1}) = q_0 + q_1q^{-1} + \cdots + q_n q^{-n} \tag{44}
\]
is the weighting polynomial for the reference signal \( w(k) \), and

\[
R(q^{-1}) = r_0 + r_1q^{-1} + \cdots + r_n q^{-n} \tag{45}
\]
is the weighting polynomial for the control input \( u(k) \). The value of an element in a particular weighting polynomial determines the importance of that variable at the corresponding point in time relative to current time \( k \). Several examples will illustrate some of the possibilities of this type of performance index.

**Example 2:** Assume that the weighting polynomials are

\[
P(q^{-1}) = 1, \quad Q(q^{-1}) = 0, \quad R(q^{-1}) = r_0 \tag{46}
\]

This makes the performance index become

\[
J = y^2(k + d) + r_0^2u^2(k) \tag{47}
\]

This is a form of quadratic performance index whose goal is to make the output small in magnitude without using excessive control energy. However, no averaging is being done as in the form of quadratic index normally associated with the Linear Quadratic Regulator (LQR) in state space form. The value of \( u(k) \) at each value of \( k \) is computed so as to minimize \( 47 \). This is a modified form of the 1-step controller mentioned previously in which some control weighting is allowed.

**Example 3:** Now assume that the weighting polynomials are

\[
P(q^{-1}) = 1, \quad Q(q^{-1}) = 1, \quad R(q^{-1}) = 0 \tag{48}
\]

With these polynomials, the performance index is

\[
J = [y(k + d) - w(k)]^2 \tag{49}
\]

Since the purpose of the control is to minimize \( J \), it is apparent that \( w(k) \) is the desired value of \( y(k + d) \), that is, \( w(k) = y^*(k + d) \). This is the performance index that leads to the 1-step control algorithm of (41). In the context of stochastic control, this is known as minimum variance control \([7],[8]\) since the variance of the error \( y(k + d) - y^*(k + d) \) is minimized. Notice that there is no term involving \( u(k) \) in the PI; therefore, it should be no surprise that very large control efforts may be required as a result of using this particular performance index.

**Example 4:** For weighting polynomials

\[
P(q^{-1}) = 1, \quad Q(q^{-1}) = 1, \quad R(q^{-1}) = r_0 (1 - q^{-1}) \tag{50}
\]
the performance index becomes

\[ J = [y(k + d) - w(k)]^2 + r_0^2 |u(k) - u(k - 1)|^2 \]  \hspace{1cm} (51)

With this PI, the output error is penalized through the first term on the right side of (51). The second term in the PI penalizes the change in value of the control signal from one time step to the next. This would be useful in preventing large changes in actuator position in one time step. For example, consider the effect on the control input if \( y^*(k + d) = +10 \) volts and \( y^*(k + d + 1) = -10 \) volts if the PI in (47) were used. Even though the magnitude of the control might not be excessive with that PI, the incremental change in control could still be large if the control value changed sign. The PI in (51) tends to limit the incremental control magnitude; hence it acts as a form of rate limiting.

\textbf{Example 5:} For the last example, assume that \( R(q^{-1}) = 0 \) and that \( P(q^{-1}) \) and \( Q(q^{-1}) \) are arbitrary (nonzero) polynomials with the degree of \( Q(q^{-1}) \) not exceeding the degree of \( P(q^{-1}) \). The performance index now is

\[ J = [P(q^{-1}) y(k + d) - Q(q^{-1}) w(k)]^2 \]  \hspace{1cm} (52)

The value of \( J \) can be made zero by choosing the control so that

\[ y(k + d) = \left[ \frac{Q(q^{-1})}{P(q^{-1})} \right] w(k) \]  \hspace{1cm} (53)

Thus, the actual system output \( y(k + d) \) is forced to follow the output of a model, described by the \( P(q^{-1}) \) and \( Q(q^{-1}) \) polynomials, when that model is driven by the reference input \( w(k) \). With this PI, \( w(k) \neq y^*(k + d) \); however, the output of the reference model given by the right side of (53) is the desired output. In any actual case of control design, the denominator polynomial for this \textit{model-following control} would be selected to be asymptotically stable.

\textit{B. The Optimal Control Law}

After using the division algorithm, the system model is given by (38). To determine the optimal control law, the right side of (38) is substituted for \( y(k + d) \) in the performance index (42). This gives us

\[ J = [P(q^{-1}) F(q^{-1}) B'(q^{-1}) u(k) + P(q^{-1}) G(q^{-1}) y(k) - Q(q^{-1}) w(k)]^2 \]  \hspace{1cm} (54)

If \( u(k) \) is unconstrained in magnitude, the optimal control can be determined by taking the partial derivative of \( J \) with respect to the variable \( u(k) \) and setting it equal to 0. Remember that we are finding the value of control input at the single point in time \( k \), that is, only a single term in an expression like \( M(q^{-1}) u(k) \). The partial derivative of \( J \) with respect to \( u(k) \) is
\[
\frac{\partial J}{\partial u(k)} = 2b_0 \left[ P(q^{-1}) F(q^{-1}) B'(q^{-1}) u(k) + P(q^{-1}) G(q^{-1}) y(k) - Q(q^{-1}) w(k) \right] + 2r_0 R(q^{-1}) u(k) = 0
\] (55)

recalling that the \( P(q^{-1}) \) and \( F(q^{-1}) \) polynomials are monic. Collecting terms in \( u(k) \) provides the following expression for the optimal control law:

\[
\left[ P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) \right] u^*(k) = -P(q^{-1}) G(q^{-1}) y(k) + Q(q^{-1}) w(k)
\] (56)

Thus, the optimal control has a negative feedback term from \( y(k) \) plus a feedforward term (outside the loop) from the external input \( w(k) \). Figure 1 illustrates this control structure in transfer function form. Making the following definitions

\[
M(q^{-1}) = P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) \] (57)

\[
N(q^{-1}) = P(q^{-1}) G(q^{-1}) \] (58)

where the degree of \( M(q^{-1}) \) is \( n_4 = (n_1 + d - 1 + m) \) and the degree of \( N(q^{-1}) \) is \( n_5 = (n_1 + n - 1) \), the control law can be written as

---

**Fig. 1.** Structure for discrete-time optimal control with polynomial plant model and performance index.
The above expression for the degree of $M(q^{-1})$ assumes that the degree of $R(q^{-1})$ is not greater than the sum of the degrees of $P(q^{-1})$, $F(q^{-1})$, and $B'(q^{-1})$. The equation to be solved recursively to determine the value of $u(k)$ at each time step is

$$u^*(k) = (1/m_0)[-m_1u(k - 1) - m_2u(k - 2) - \cdots - m_n u(k - n_4)$$

$$- n_0y(k) - n_1y(k - 1) - n_2y(k - 2) - \cdots - n_n y(k - n_5)$$

$$+ q_0w(k) + q_1w(k - 1) + q_2w(k - 2) + \cdots + q_n w(k - n_2)]$$

Assuming that the needed values of the reference signal $w(k - i)$ are known at time $k$, the procedure to apply this control algorithm is essentially the same as discussed previously, namely:

- Measure the output signal $y(k)$;
- Access the value for the reference signal $w(k)$;
- Compute the control signal $u^*(k)$ from (60);
- Apply that $u^*(k)$ to control the system until the next time step;
- Shift the input, output, and reference variables backwards one position to be ready for the next set of computations.

IV. THE CLOSED-LOOP SYSTEM

To determine the model for the closed-loop system, (56) can be solved for $u^*(k)$, and that expression can be substituted for $u(k)$ in the open-loop system model given by (13). Several steps in the derivation of the closed-loop system model are shown below.

$$A(q^{-1}) y(k + d) = B'(q^{-1}) \cdot -P(q^{-1}) G(q^{-1}) y(k) + Q(q^{-1}) w(k)$$

$$\frac{-P(q^{-1}) G(q^{-1}) y(k) + Q(q^{-1}) w(k)}{P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1})}$$

$$\{[P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1})] A(q^{-1}) + q^{-d} B'(q^{-1}) P(q^{-1}) G(q^{-1})\} y(k + d)$$

$$= B'(q^{-1}) Q(q^{-1}) w(k)$$

$$\{P(q^{-1}) B'(q^{-1}) [A(q^{-1}) F(q^{-1}) + q^{-d} G(q^{-1})] + (r_0/b_0) R(q^{-1}) A(q^{-1})\} y(k + d)$$

$$= B'(q^{-1}) Q(q^{-1}) w(k)$$

and finally

$$M(q^{-1}) u^*(k) = -N(q^{-1}) y(k) + Q(q^{-1}) w(k)$$

(59)
\[ \left[ P(q^{-1})B'(q^{-1}) + (r_0/b_0)R(q^{-1})A(q^{-1}) \right] g(k + d) = B'(q^{-1})Q(q^{-1})w(k) \] (64)

where use of the division algorithm from (32) has been used in going from (63) to (64). Note the use of \( q^{-d} \) in (62) to allow the collection of terms in the output variable.

Writing (64) as

\[ A_{cl}(q^{-1})g(k + d) = B_{cl}(q^{-1})w(k) \] (65)

with

\[ \deg[A_{cl}(q^{-1})] = \max[(n_1 + m), (n_3 + n)] = n_A \] (66)

\[ \deg[B_{cl}(q^{-1})] = m + n_2 = n_B \] (67)

the closed-loop transfer function can be found by taking the Z-transform of (65).

\[ \frac{Y(z)}{W(z)} = \frac{B_{cl}(z)}{z^{(d-n_A+n_B)}A_{cl}(z)} = \frac{c_0z^{n_B} + c_1z^{n_B-1} + \cdots + c_n}{z^{(d-n_A+n_B)}(d_0z^{n_A} + d_1z^{n_A-1} + \cdots + d_n)} \] (68)

In comparing (64) with (65), you can see that \( B_{cl}(q^{-1}) = B'(q^{-1})Q(q^{-1}) \). Thus, the open-loop system zeros are also closed-loop zeros. Another very important point to note is that if \( r_0 = 0 \), then \( A_{cl}(q^{-1}) = P(q^{-1})B'(q^{-1}) \), and the open-loop zeros are also closed-loop poles. The tendency here is to cancel the \( B'(q^{-1}) \) polynomial from both sides of the equation. However, this can be done safely only when \( B'(q^{-1}) \) is asymptotically stable (and its roots are well damped). In that case, \( B'(q^{-1}) \) can be canceled, and the performance of the system is governed by (53). If the open-loop system is non-minimum-phase (\( B'(q^{-1}) \) has zeros outside the unit circle), then the closed-loop system will be unstable whenever \( r_0 = 0 \). Thus, for a non-minimum-phase system, it is necessary to include the \( R(q^{-1}) \) polynomial with \( r_0 \neq 0 \) in order to achieve closed-loop stability.

One factor that may not be realized is that when the discrete-time model is derived from a continuous-time system, the discrete-time system may be non-minimum-phase even when the continuous-time system is minimum phase \([9]\). Depending on the relative order of the continuous-time transfer function and the sampling period used to produce the discrete-time model, left-half plane continuous-time zeros can become discrete-time zeros outside the unit circle. Therefore, the exact location of the discrete-time poles and zeros should be checked before choosing a performance index.

**V. Example Problem**

A simple example is presented to illustrate different performances that can be achieved by modifying the performance index. None of the PIs in this example is necessarily a good one; that is a judgment choice of the designer and depends on the specific application. The open-loop system model is
\[(1 - 9q^{-1} + 26q^{-2} - 24q^{-3}) y(k + 2) = (1 - 0.25q^{-1}) u(k) \]  \hspace{1cm} (69)

so the open-loop poles are at \( z = \{2, 3, 4\} \). The open-loop zero is at \( z = 0.25 \), and the control delay is \( d = 2 \) time steps. The reference signal \( w(k) \) will be a square wave of unit amplitude and period 50 time steps.

A. 1-Step Control

The first PI is the one giving 1-step control, specified by (49). This PI should force the output \( y(k + 2) \) to be exactly equal to the reference signal \( w(k) \). Since the open-loop system does not have any zeros on or outside the unit circle, stability is not affected by having \( R(q^{-1}) = 0 \). Figure 2 shows the output, reference, and control input \( u(k) \) in response to this PI. The output follows the reference input exactly, with a 2-step time delay. Note the amplitudes of the control input, and the large changes in those amplitudes between consecutive points in time, that are required to achieve this exact tracking. The polynomials for the control law and the closed-loop system are

\[
M(q^{-1}) = P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) = 1 + 8.75q^{-1} - 2.25q^{-2}
\]  \hspace{1cm} (70)

\[
N(q^{-1}) = P(q^{-1}) G(q^{-1}) = 55 - 210q^{-1} + 216q^{-2}
\]  \hspace{1cm} (71)

\[
A_{cl}(q^{-1}) = P(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) A(q^{-1}) = 1 - 0.25q^{-1}
\]  \hspace{1cm} (72)

\[
B_{cl}(q^{-1}) = B'(q^{-1}) Q(q^{-1}) = 1 - 0.25q^{-1}
\]  \hspace{1cm} (73)

After canceling the pole-zero pair at \( z = 0.25 \), the remaining 4 closed-loop poles are at the origin. This will always be the case for 1-step control. Note that in the backward shift format, the poles at the origin are not evident. They would be shown explicitly in the transfer function.

B. Model Following Control

The second PI is the one giving model following control, characterized by \( R(q^{-1}) = 0 \), and the \( P(q^{-1}) \) and \( Q(q^{-1}) \) polynomials specifying the reference model that the actual system should track. The reference model is given by

\[
\frac{Q(q^{-1})}{P(q^{-1})} = \frac{0.04}{1 - 1.6q^{-1} + 0.64q^{-2}}
\]  \hspace{1cm} (74)

This represents a second-order, critically damped system, with the poles at \( z = 0.8 \). The value of \( Q(q^{-1}) \) provides unity gain between the input and output in steady state. Figure 3 shows the output, reference, and control input with this PI. The output obviously responds much more sluggishly than with the previous PI, never quite reaching the final value before the reference input changes again. On the other
Fig. 2. Response of the system using 1-step control.
hand, the maximum magnitudes for the control input, and their rates of change, are much less than with the previous PI. The polynomials for the control law and the closed-loop system are

\[
M(q^{-1}) = P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) = 1 + 7.15q^{-1} - 15.61q^{-2} + 9.2q^{-3} - 1.44q^{-4}
\]

(75)

\[
N(q^{-1}) = P(q^{-1}) G(q^{-1}) = 55 - 298q^{-1} + 587.2q^{-2} - 480q^{-3} + 138.24q^{-4}
\]

(76)

\[
A_{cl}(q^{-1}) = P(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) A(q^{-1}) = 1 - 1.85q^{-1} + 1.04q^{-2} - 0.16q^{-3}
\]

(77)

\[
B_{cl}(q^{-1}) = B'(q^{-1}) Q(q^{-1}) = 0.04 (1 - 0.25q^{-1})
\]

(78)

After the pole-zero pair at \(z = 0.25\) is cancelled, two of the closed-loop poles are at \(z = 0.8\), as specified by the model, and the remaining 4 poles are at the origin.

C. Modified 1-Step Control

The last PI produces the modified 1-step controller. The weighting polynomials in the PI are

\[
P(q^{-1}) = 1, \quad Q(q^{-1}) = 0.5, \quad R(q^{-1}) = 0.1
\]

(79)

The value of \(Q(q^{-1})\) produces unity gain in steady state. Figure 4 shows the output, reference, and control input using these values. Exact matching of the output with the reference signal is not achieved, but the square wave characteristics of the reference signal are apparent in the output. In comparing the control input for this PI with that in Fig. 2, it can be seen that including an input weighting polynomial \(R(q^{-1})\) has slightly decreased the maximum magnitude of the control input. A further reduction could be made by increasing the value of \(r_0\), the leading coefficient in \(R(q^{-1})\). However, closed-loop stability becomes an issue if \(r_0\) is increased too much. The polynomials for the control law and the closed-loop system are

\[
M(q^{-1}) = P(q^{-1}) F(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) = 1.01 + 8.75q^{-1} - 2.25q^{-2}
\]

(80)

\[
N(q^{-1}) = P(q^{-1}) G(q^{-1}) = 55 - 210q^{-1} + 216q^{-2}
\]

(81)

\[
A_{cl}(q^{-1}) = P(q^{-1}) B'(q^{-1}) + (r_0/b_0) R(q^{-1}) A(q^{-1}) = 1 - 0.34q^{-1} + 0.26q^{-2} - 0.24q^{-3}
\]

(82)

\[
B_{cl}(q^{-1}) = B'(q^{-1}) Q(q^{-1}) = 0.5 (1 - 0.25q^{-1})
\]

(83)

and the closed-loop poles are at \(z = 0, 0, -0.125 \pm j0.624, 0.587\). The oscillation in the output signal is caused by the pair of complex closed-loop poles. Since \(R(q^{-1}) \neq 0\), the open-loop zero at \(z = 0.25\) does not appear as a closed-loop pole, although it still is a closed-loop zero.
Fig. 3. Response of the system using model-following control.
Fig. 4. Response of the system using modified 1-step control.
VI. Summary and Concluding Remarks

In this presentation we have considered discrete-time systems in terms on input-output models, such as given in (13). A quadratic performance index, (42), was introduced, the polynomials of which could be chosen to represent different performance goals for the system. In order to use this PI, the system model was manipulated through the use of the division algorithm, shown in (32), into a predictor format given in (38). With this PI, the optimal control law is always given by (56), and the recursive implementation of the control is given by (60).

Here we have only considered single-input, single-output systems. The techniques can be extended to multi-input, multi-output systems as well. The polynomials in (13) now become matrices of polynomials. For a system with \( p \) inputs and \( l \) outputs, \( A (q^{-1}) \) is \( p \times p \) and \( B (q^{-1}) \) is \( p \times l \), having the forms

\[
A (q^{-1}) = \begin{bmatrix}
    A_{11} (q^{-1}) & A_{12} (q^{-1}) & \cdots & A_{1p} (q^{-1}) \\
    \vdots & \vdots & \cdots & \vdots \\
    A_{p1} (q^{-1}) & A_{p2} (q^{-1}) & \cdots & A_{pp} (q^{-1})
\end{bmatrix}
\]

\[
B (q^{-1}) = \begin{bmatrix}
    B_{11} (q^{-1}) & B_{12} (q^{-1}) & \cdots & B_{1l} (q^{-1}) \\
    \vdots & \vdots & \cdots & \vdots \\
    B_{p1} (q^{-1}) & B_{p2} (q^{-1}) & \cdots & B_{pl} (q^{-1})
\end{bmatrix}
\]

where the \( A_{ii} (q^{-1}) \) polynomials are monic, and the \( A_{ij} (q^{-1}) \) polynomials have the constant term equal to 0 for \( i \neq j \).

The locations of the open-loop zeros must be checked before the choice of performance index is finalized. If the discrete-time system is non-minimum-phase, then the \( R (q^{-1}) \) polynomial must be included in the PI. This is a necessary but not sufficient condition for closed-loop stability. You still have to find a stabilizing \( R (q^{-1}) \). Thus, the polynomial approach to control design using a quadratic PI does not have the same guaranteed stability properties of the state space Linear Quadratic Regulator (LQR) or Linear Quadratic Gaussian (LQG) methods.

Always remember that unstable poles (zeros) in the controller must never cancel unstable zeros (poles) in the plant. The real system will still have unstable modes that will either be not observable at the output or not controllable from the input. If the closed-loop system is internally stable, then stability is maintained no matter where an external signal enters the system and no matter what signal is regarded as the output. A necessary condition for internal stability is that there be no unstable pole/zero cancellations between blocks.

The input-output approach presented here is an alternative to the state space representation that is often described. The polynomial approach is particularly convenient when the model has been developed.
from experimental data based on measurements of the input and output. In that case, an input-output model is most natural, since the internal state variables would not be directly available from the data.

References