Spectral analysis using the DFT

System for Fourier analysis of CT signal $s_c(t)$:

Windowing operation: $v[n] = w[n]x[n]$

- selects a portion of signal for processing
- necessary since DFT only works on finite-length sequences
- general intuition: windowing in time smears the spectrum, *i.e.*, (convolve $X(e^{j\theta})$ with $W(e^{j\theta})$
- two effects controlled by window
  - resolution – controlled by mainlobe width
    How close can two sinusoids be and still be resolvable?
  - leakage – controlled by sidelobe height
    Loud tone can mask a quiet tone.

Spectral sampling (*i.e.*, the number of DFT samples we take) affects our ability to visualize the underlying spectrum of the windowed signal.
Application example

Consider the CT signal $x(t)$:

Analyze using 2 sec rectangular window and 1024-pt transform:

Zoom in on -15 to +15 Hz range:
Application example

Let low frequency sinusoid equal to 1 Hz (vs. 3 Hz): resolved!

Now lower it to 0.5 Hz: not resolved!

Reasoning:
We have 2 second window and sample rate is 100 Hz

2 second window contains (2 seconds)(100 samples/second)=200 samples

Approximate resolution of rectangular window (mainlobe width in Hz):

\[
\frac{4\pi}{\text{window length}} = 4\pi \times 200 \quad \text{in Hz :} \quad \frac{4\pi}{2\pi \times \text{(length)}} f_s = \frac{200}{200} = 1\text{Hz}
\]
Windows for spectral analysis

Windows defined for $0 \leq n \leq N - 1$:

Rectangular: $w[n] = 1$

Bartlett: $w[n] = 1 - \frac{|2n-N+1|}{N+1}$

Hanning: $w[n] = 0.5 \left[ 1 - \cos \left( \frac{2\pi n}{N-1} \right) \right]$

Hamming: $w[n] = 0.54 - 0.46 \cos \left( \frac{2\pi n}{N-1} \right)$

Blackman: $w[n] = 0.42 - 0.5 \cos \left( \frac{2\pi n}{N-1} \right) + 0.08 \cos \left( \frac{4\pi n}{N-1} \right)$
Transforms of the window functions

Windows normalized so that peak is 0 dB:
## Window characteristics

Comparison of commonly-used windows (from Oppenheim/Schafer):

<table>
<thead>
<tr>
<th>Window type</th>
<th>Peak sidelobe (dB)</th>
<th>Approximate mainlobe width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>-13</td>
<td>$4\pi/N$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>-25</td>
<td>$8\pi/(N - 1)$</td>
</tr>
<tr>
<td>Hanning</td>
<td>-31</td>
<td>$8\pi/(N - 1)$</td>
</tr>
<tr>
<td>Hamming</td>
<td>-41</td>
<td>$8\pi/(N - 1)$</td>
</tr>
<tr>
<td>Blackman</td>
<td>-57</td>
<td>$12\pi/(N - 1)$</td>
</tr>
</tbody>
</table>

- peak sidelobe is relative to peak of the window transform
- width of mainlobe refers to distance between zero crossings