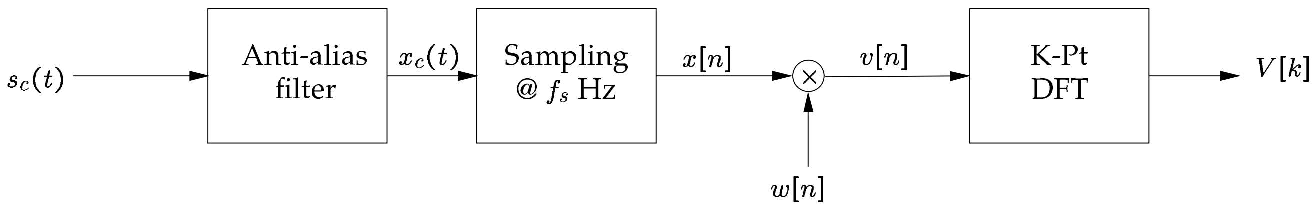
Spectral analysis using the DFT

System for Fourier analysis of CT signal $s_c(t)$:



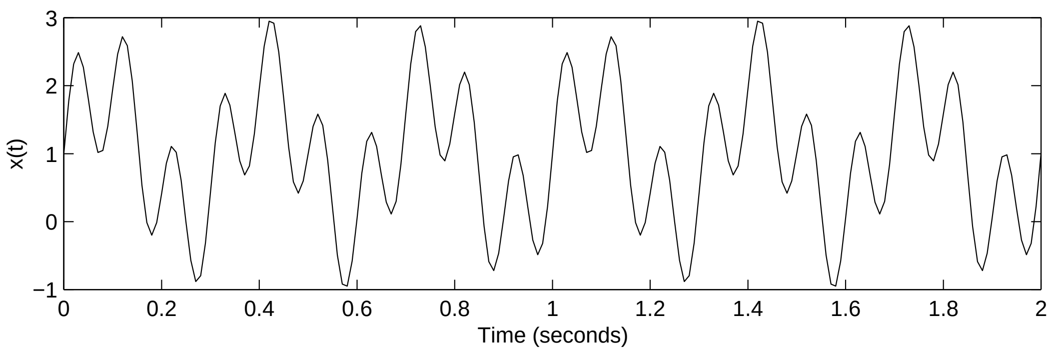
Windowing operation: $v[n] = w[n]x[n]$

- selects a portion of signal for processing
- necessary since DFT only works on finite-length sequences
- general intuition: windowing in time smears the spectrum, *i.e.*, (convolve $X(e^{j\theta})$ with $W(e^{j\theta})$)
- two effects controlled by window
  - resolution – controlled by mainlobe width
    How close can two sinusoids be and still be resolvable?
  - leakage – controlled by sidelobe height
    Loud tone can mask a quiet tone.

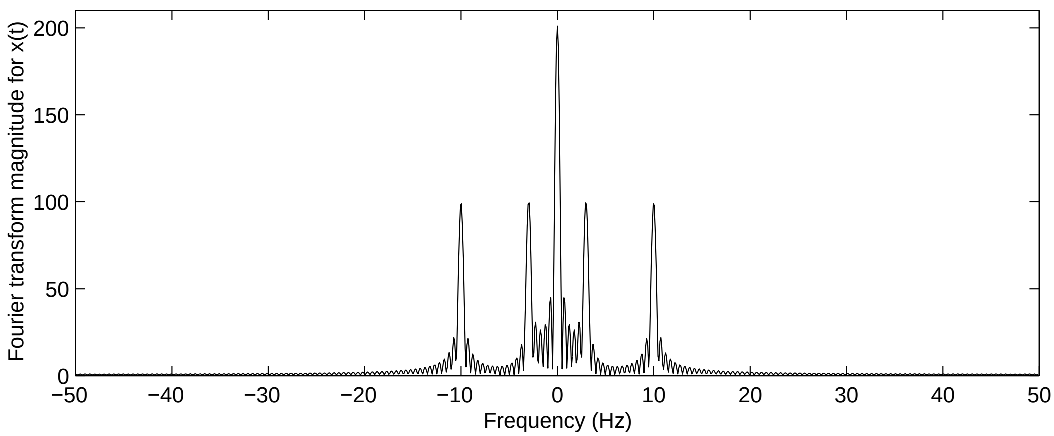
Spectral sampling (*i.e.*, the number of DFT samples we take) affects our ability to visualize the underlying spectrum of the windowed signal

Application example

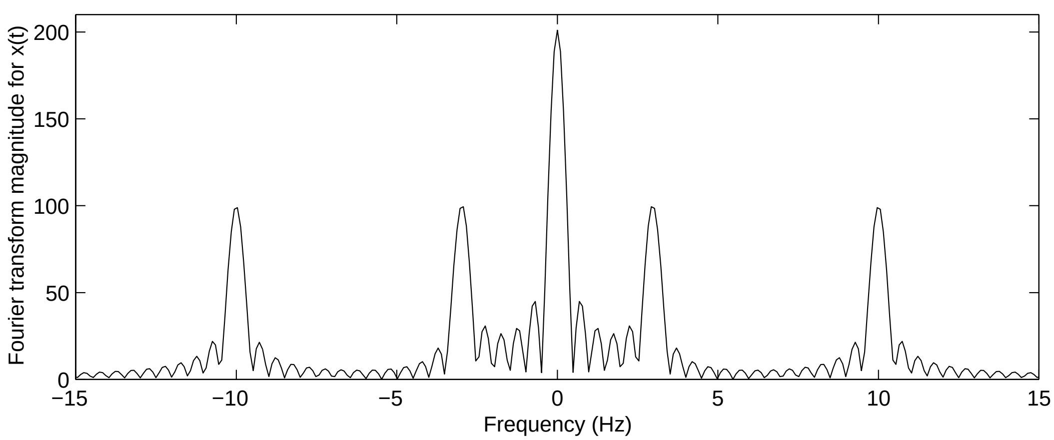
Consider the CT signal $x(t)$:



Analyze using 2 sec rectangular window and 1024-pt transform:

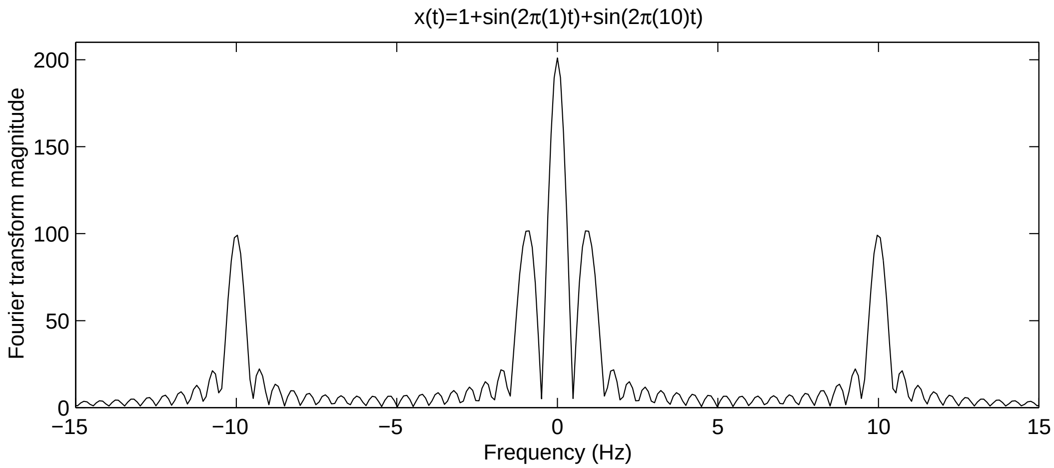


Zoom in on -15 to +15 Hz range:

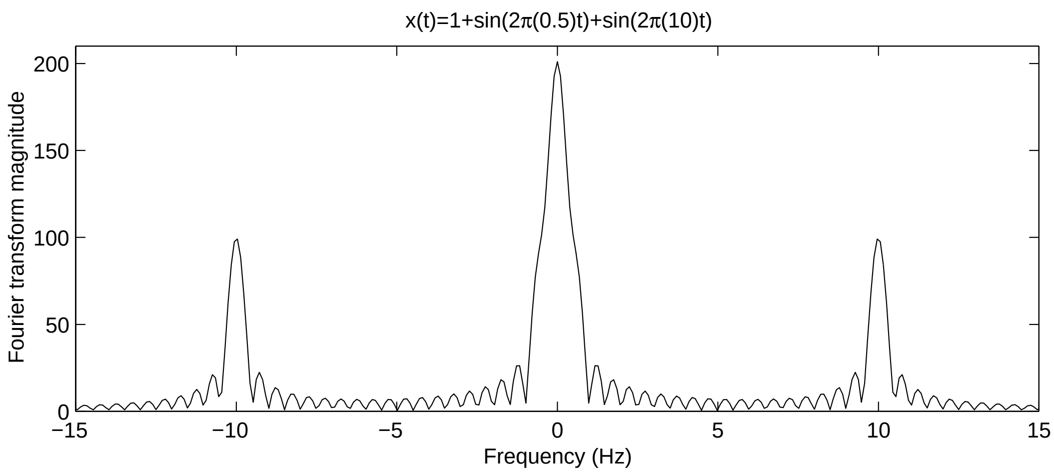


Application example

Let low frequency sinusoid equal to 1 Hz (vs. 3 Hz): resolved!



Now lower it to 0.5 Hz: not resolved!



Reasoning:

We have 2 second window and sample rate is 100 Hz

2 second window contains (2 seconds)(100 samples/second)=200 samples

Approximate resolution of rectangular window (mainlobe width in Hz):

$$\frac{4\pi}{\text{window length}} = 4\pi 200 \longrightarrow \text{in Hz : } \frac{4\pi}{2\pi(\text{length})} f_s = \frac{200}{200} = 1\text{Hz}$$

Windows for spectral analysis

Windows defined for $0 \leq n \leq N-1$:

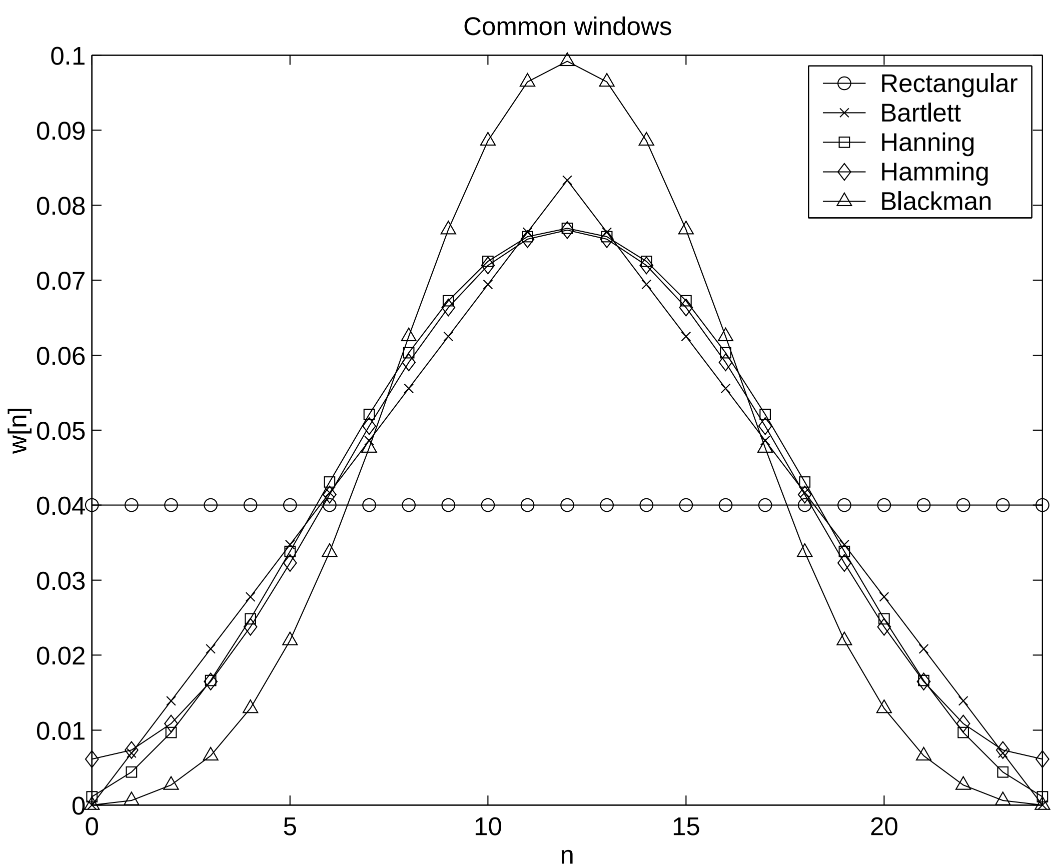
Rectangular: $w[n] = 1$

Bartlett: $w[n] = 1 - \frac{|2n-N+1|}{N+1}$

Hanning: $w[n] = 0.5\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]$

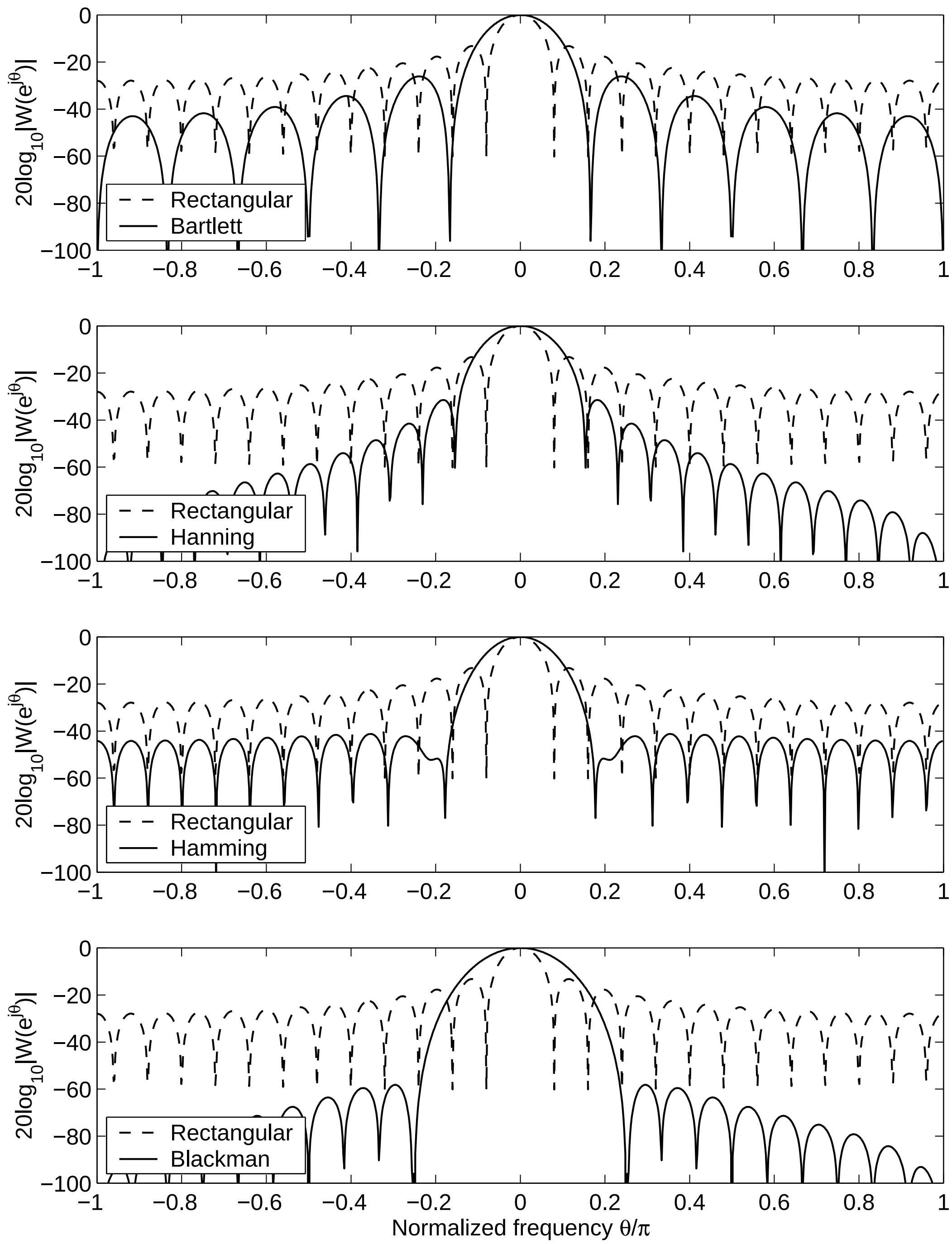
Hamming: $w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$

Blackman: $w[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$



Transforms of the window functions

Windows normalized so that peak is 0 dB:



Window characteristics

Comparison of commonly-used windows (from Oppenheim/Schafer):

Window type	Peak sidelobe (dB)	Approximate mainlobe width
Rectangular	-13	$4\pi/N$
Bartlett	-25	$8\pi/(N-1)$
Hanning	-31	$8\pi/(N-1)$
Hamming	-41	$8\pi/(N-1)$
Blackman	-57	$12\pi/(N-1)$

- peak sidelobe is relative to peak of the window transform
- width of mainlobe refers to distance between zero crossings