

Computing the DFT

$$\text{DFT: } X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\text{IDFT: } x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

- DFT is N equally-spaced samples of DT Fourier transform

$$\omega_k = \frac{2\pi k}{N} \quad 0 \leq \omega_k < 2\pi$$

- Note similarity of DFT and IDFT
 - given DFT implementation \longrightarrow easy to get IDFT implementation

\Rightarrow Computational complexity is the key implementation issue

Computational complexity

Measures of complexity:

- Number of multiplies and adds
- Chip area
- Power requirements

Direct implementation of the DFT:

$$\sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Each sample $X[k]$ requires:

N complex multiplies

$N - 1$ complex adds

Thus an N -pt DFT requires: $\mathcal{O}(N^2)$ computations

\Rightarrow For large N , computational burden is huge

Computing the DFT (continued)

We want algorithms for DFT that reduce number of computations.

These methods usually exploit symmetry/periodicity of W_N^{kn} .

Brief history:

- 1805: Gauss – origins of the fast Fourier transform
 - 1905: Runge
 - 1942: Danielson/Lanczos
 - 1965: Cooley and Tukey
 - developed fast algorithms for DFT when N is a composite number
- ← $\mathcal{O}(N \log N)$

⇒ The latter touched off a flurry of research leading to the FFT

Fast Fourier Transform

FFT is an algorithm for computing the DFT

Basic idea (Cooley & Tukey):

Split DFT into smaller and smaller transforms

If N is a power of 2: $N = 2^\nu$

- Number of computations is $\mathcal{O}(N \log_2 N)$
- Called *radix-2*

Two categories:

- Decimation-in-Frequency
 - compute even/odd frequency samples separately
- Decimation-in-Time
 - compute using even/odd time samples separately

Decimation in frequency algorithm

Suppose we have a 1024-point signal

- What length DFT is sufficient to represent this signal? **1024 points**
- Suppose only even DFT samples are needed. How can we get them?
Undersampling in frequency \Rightarrow time aliasing
Thus: time-alias and compute 512-point transform to get even samples
- Suppose only odd DFT samples are needed. How can we get them?
 - Multiply time sequence by complex exponential
 \Rightarrow this produces shift in frequency domain
 - **Time alias the new sequence**
 - **Take 512-point transform**

Decimation in frequency: even samples

Assume N is a power of 2

To get even DFT samples:

$$\begin{aligned}
 X[2r] &= \sum_{n=0}^{N-1} x[n] W_N^{n(2r)} & r = 0, \dots, \frac{N}{2} - 1 \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2rn} \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{2rn} \underbrace{W_N^{2r\left(\frac{N}{2}\right)}}_{W_N^{rN} = e^{-j\frac{2\pi}{N}rN} = e^{-j2\pi r} = 1}
 \end{aligned}$$

Even samples: $X[2r] = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x\left[n + \frac{N}{2}\right] \right) W_{\frac{N}{2}}^{rn}}_{\frac{N}{2}\text{-pt DFT of time-aliased sequence}}$

Decimation in frequency: odd DFT samples

Odd DFT samples:

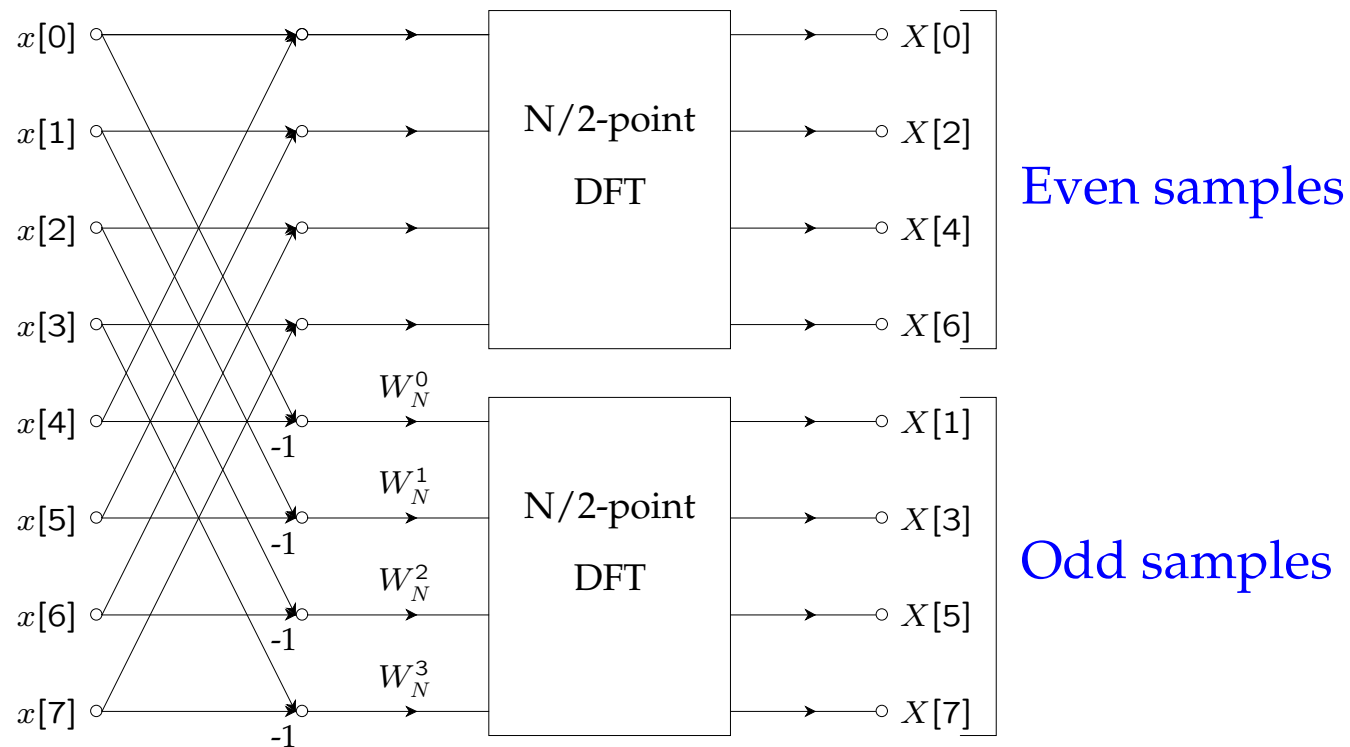
$$X[2r + 1] = \sum_{n=0}^{\frac{N}{2}-1} \left[x[n] W_N^n + x \left[n + \frac{N}{2} \right] W_N^n \underbrace{W_{\frac{N}{2}}^{\frac{N}{2}}}_{e^{-j\pi}} \right] W_{\frac{N}{2}}^{nr}$$

$$X[2r + 1] = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} \left[x[n] - x \left[n + \frac{N}{2} \right] \right] W_N^n W_{\frac{N}{2}}^{nr}}_{\frac{N}{2}\text{-pt DFT of time-aliased \& modulated sequence}}$$

- $x[n] - x \left[n + \frac{N}{2} \right]$ is another form of time-aliasing
- the W_N^n term is the modulation

8-point DFT example \Rightarrow

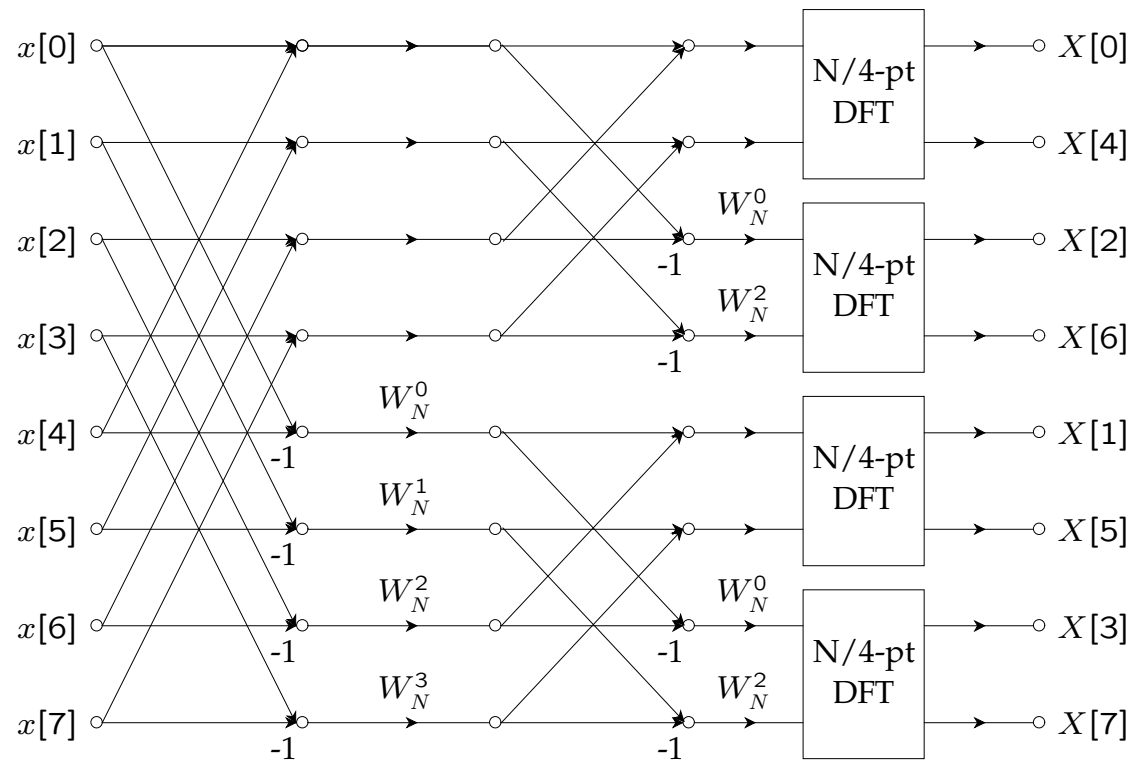
Decimation in frequency example: 8-point DFT



- W_N terms are called twiddle factors

Decompose $\frac{N}{2}$ DFT's into smaller DFT's \Rightarrow

Decimation in frequency example: 8-point DFT



- $\frac{N}{4}$ -point DFT's are 2-point DFT's if $N = 8$

What does a 2-point DFT look like? \Rightarrow

2-point DFT

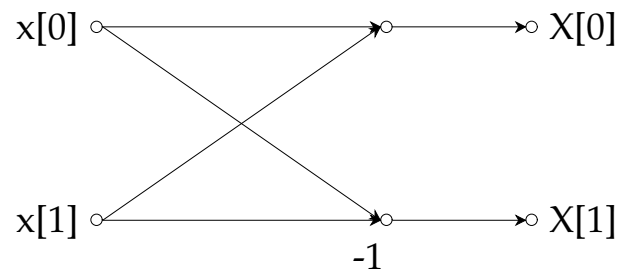
$$X[k] = \sum_{n=0}^1 x[n]e^{-j\frac{2\pi}{2}nk} = x[0]e^{-j0} + x[1]e^{-j\pi k}$$

Thus the 2-point DFT is trivial: just add and subtract!

$$X[0] = x[0] + x[1]$$

$$X[1] = x[0] - x[1]$$

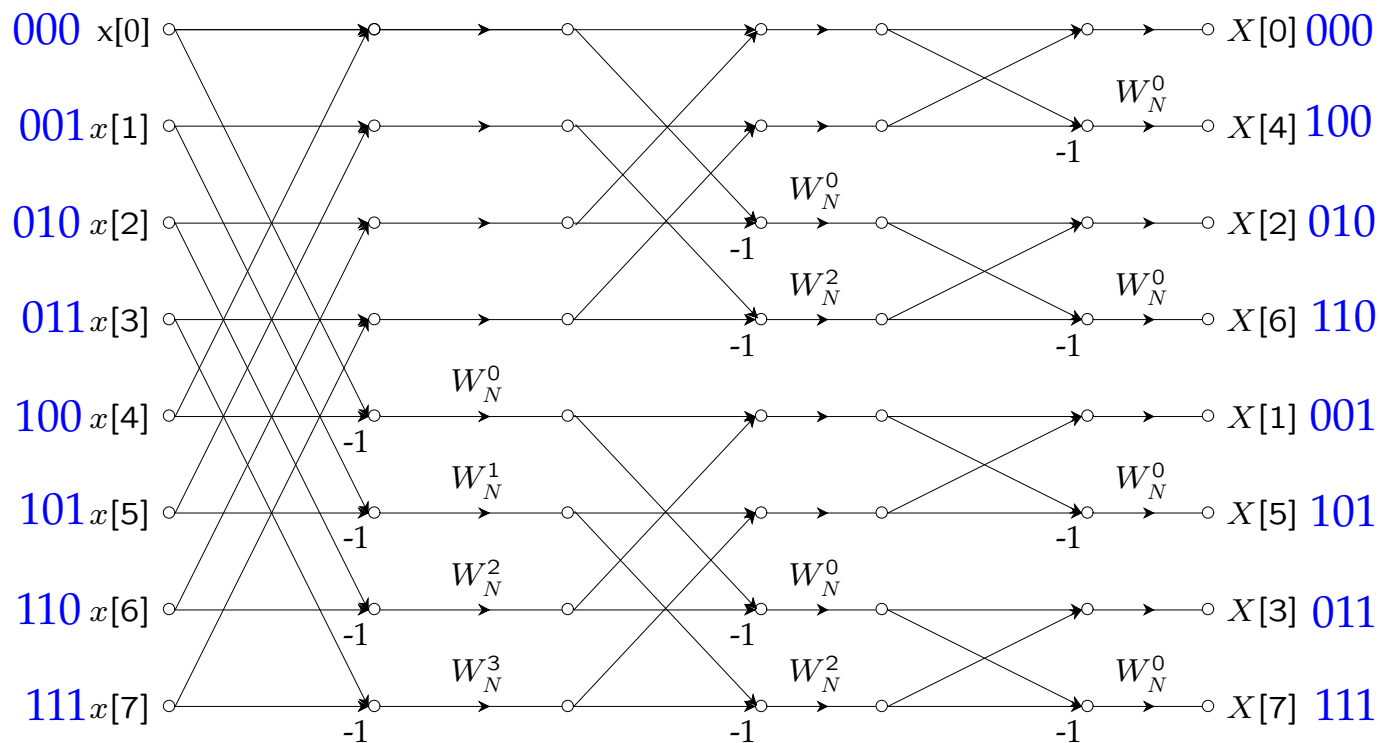
Flowgraph for 2-point DFT:



- Known as a butterfly calculation

Insert butterfly for 2-pt DFT into structure \Rightarrow

Decimation in frequency example: 8-point DFT



- Butterfly structure
- Input in normal order; output is bit-reversed

FFT Computations

First stage:

$\frac{N}{2}$ complex multiplies

N complex adds

Can decompose into $\log_2 N$ stages. Final stage has 2-pt DFT's.

Comparison: decimation-in-frequency vs. direct

Operation	Decimation-in-frequency	Direct
complex multiplies	$\frac{N}{2} \log_2 N$	N^2
complex adds	$N \log_2 N$	$N(N - 1)$

Example: $N = 1024$ $N^2 = 1,048,576$ $N \log_2 N = 10,240$

Computations reduced by 2 orders of magnitude!

Decimation in time

Recall tranposition theorem for flowgraphs:

- interchange input and output
- reverse direction of all arrows

Transpose decimation-in-frequency flowgraph

⇒ decimation-in-time flowgraph

Alternatively, we can derive decimation-in-time algorithm ⇒

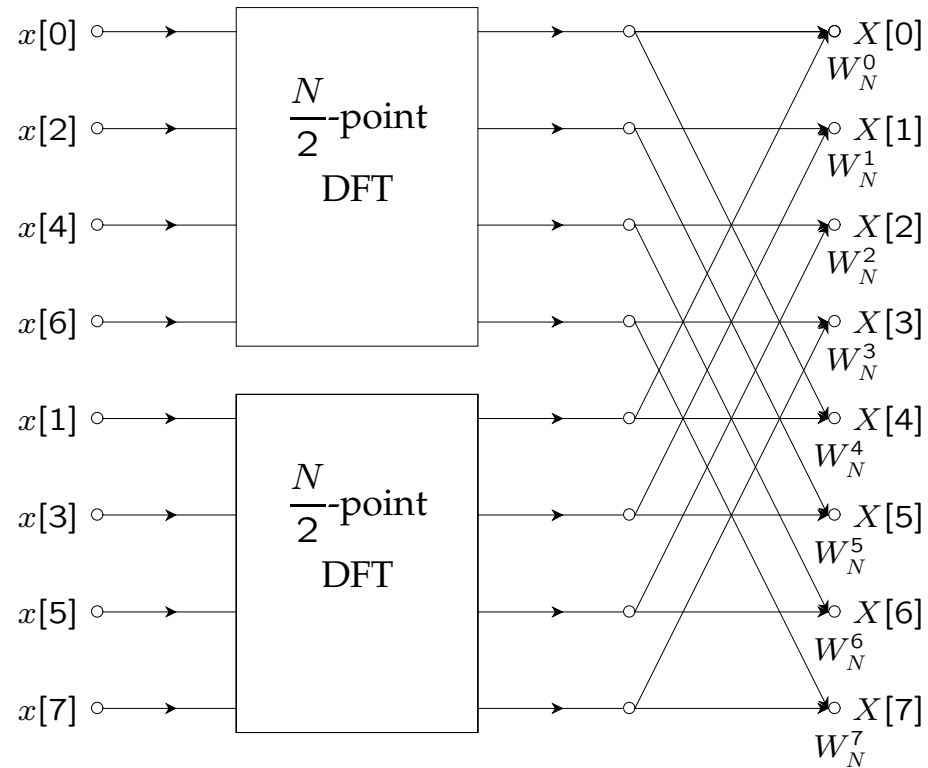
Decimation in time derivation

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{nk} \\ &= \sum_{n,\text{even}} x[n]W_N^{nk} + \sum_{n,\text{odd}} x[n]W_N^{nk} \\ &= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2rk} + \left(\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{2rk} \right) W_N^k \\ &= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{N/2}^{rk}}_{\frac{N}{2}\text{-pt DFT of even samples}} + \underbrace{\left(\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{N/2}^{rk} \right)}_{\frac{N}{2}\text{-pt DFT of odd samples}} W_N^k \end{aligned}$$

Divide into $\log_2 N$ stages as before \Rightarrow

Decimation in time structure

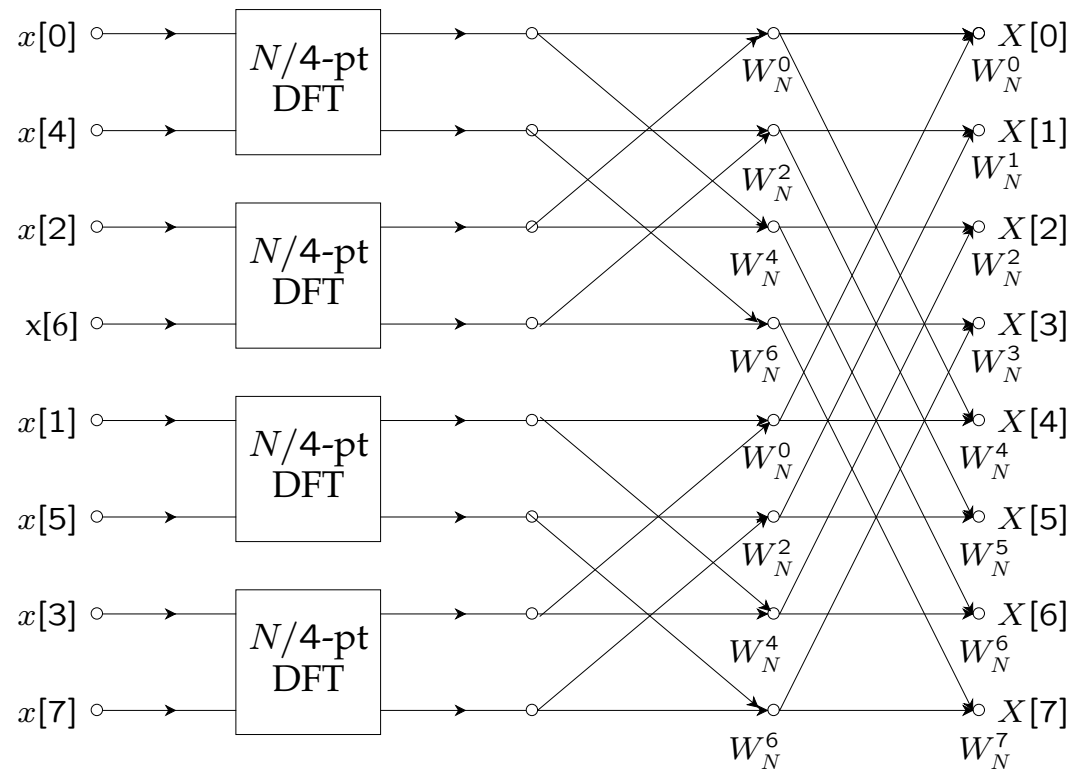
Even samples



Odd samples

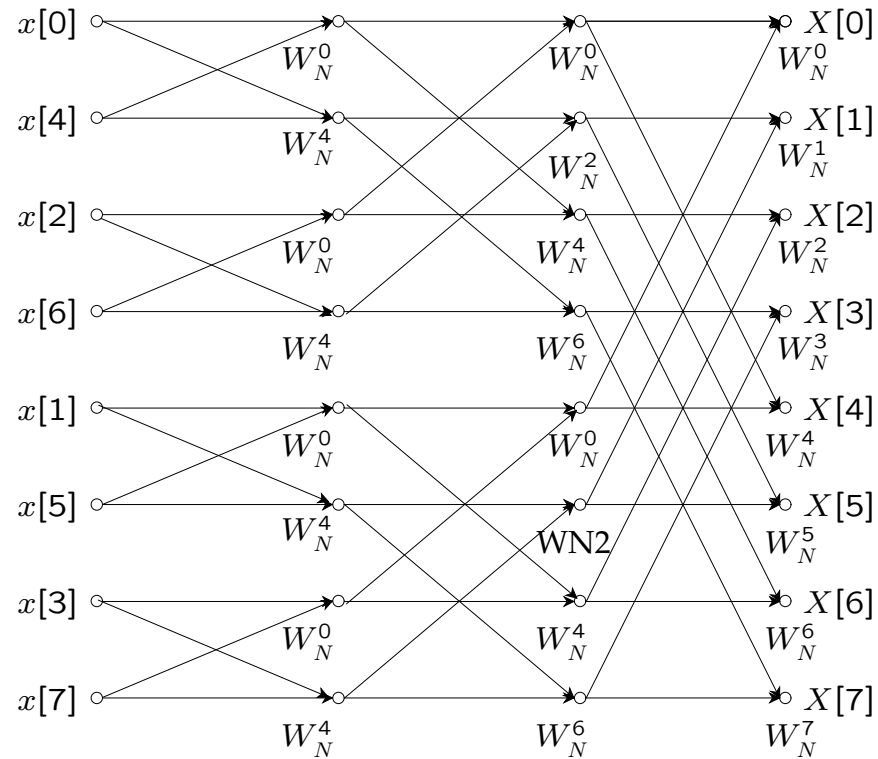
Decompose $\frac{N}{2}$ -pt DFT's \Rightarrow

Decimation in time structure



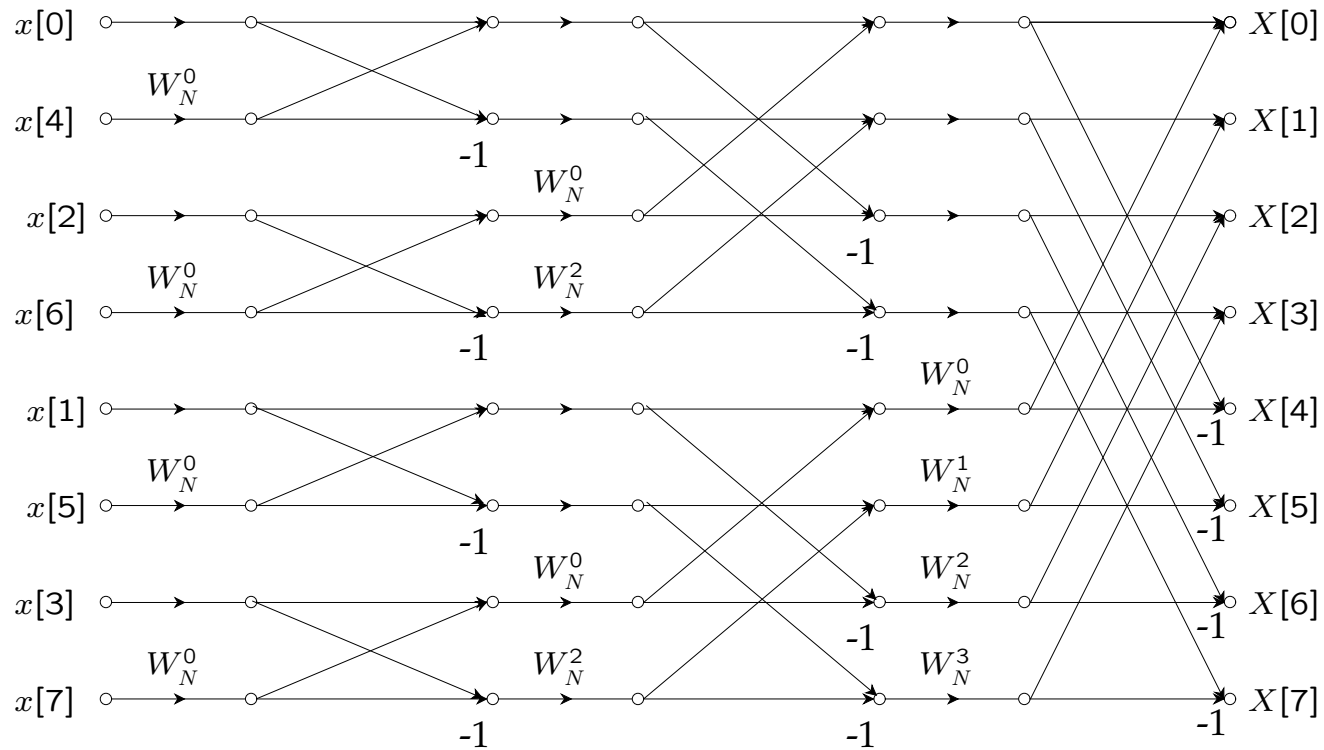
Decompose $\frac{N}{4}$ -pt DFT's \Rightarrow

Decimation in time structure



Can simplify the butterfly calculations \Rightarrow

Decimation in time structure



- Bit-reversed input; normal-order output

Many other structures possible \Rightarrow

Which implementation/flowgraph to use?

Considerations:

- In-place calculation
- Indexing strategy
- Random or sequential access memory
- Store coefficients or compute recursively

In-class exercise

Sketch the signal $x[n]$ and compute its 4-point DFT:

$$x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2] + \delta[n - 3].$$

In-class exercise

Verify that the following decimation-in-frequency flowgraph produces the same result for $x[n]$:

