Overview:

- **Motivation**
  Many applications require analyzing the frequency content of signals
  - Speech processing – study resonances of vocal tract
  - Radar – Doppler processing
  - Sonar – study channel effects on long-range transmissions

- **Basic spectral analyzer setup**
  - Windowing
  - Frequency-sampling
  - Relating DFT output to CT spectrum

- **Examples**

- **Time-dependent Fourier transform**
**Basic spectral analyzer**

\[ s_c(t) \xrightarrow{\text{Anti-Alias Filter}} x_c(t) \xrightarrow{\text{C/D (or A/D) Conversion}} x[n] \times v[n] \xrightarrow{\text{N-point DFT}} V[k] \]

- Anti-alias filtering and A/D conversion
  - for tonight assume negligible errors
- Windowing operation
  - Necessary because DFT only works on finite-length sequences.
  - Typically use same windows as for filter design
- Frequency-sampling
  - Use FFT to compute samples of the DTFT

Understanding windowing & frequency-sampling is crucial! ⇒
Windowing and frequency-sampling

What does the windowing operation do?

\[ v[n] = w[n] \cdot x[n] \xrightarrow{\mathcal{F}} V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta})W(e^{j(\omega-\theta)}) d\theta \]

- Convolution with window function smears the spectrum of \( x[n] \)
- Smoothes sharp peaks and discontinuities
Frequency-sampling

What is the output of the $N$-point DFT operation?

$$V[k] = \sum_{n=0}^{N-1} v[n] e^{-j \frac{2\pi kn}{N}} \quad k = 0, \ldots, N - 1$$

$$= V(e^{j\omega})\bigg|_{\omega = \frac{2\pi k}{N}}$$

- DFT is samples of the DTFT evenly spaced around the unit circle
Relating DFT output to the CT spectrum

Suppose $x_c(t)$ is sampled at a rate of $f_s = \frac{1}{T}$.
What CT frequencies correspond to the $N$ DFT samples?

Recall relationship between DT and CT frequencies: $\omega = \Omega T$

Thus the CT frequency associated with the $k$th bin is:

$$\Omega_k = \frac{\omega_k}{T} = \frac{2\pi k}{NT} = \frac{2\pi kf_s}{N}$$

Using this relationship, create a vector of analog frequencies to plot the spectrum against

Consider analyzing a signal consisting of 2 sinusoids $\Rightarrow$
Two sinusoid example

Consider $s_c(t)$ equal to the sum of 2 sinusoids: 

$$s_c(t) = A_0 \cos(\Omega_0 t) + A_1 \cos(\Omega_1 t)$$

The sampled signal $x[n]$ becomes:

$$x[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n) \quad \omega_0 = \Omega_0 T \quad \omega_1 = \Omega_1 T$$

The windowed signal $v[n]$ is:

$$v[n] = A_0 w[n] \cos(\omega_0 n) + A_1 w[n] \cos(\omega_1 n)$$

$$= \frac{A_0}{2} w[n] (e^{j\omega_0 n} + e^{-j\omega_0 n}) + \frac{A_1}{2} w[n] (e^{j\omega_1 n} + e^{-j\omega_1 n})$$

Taking the DTFT:

$$V(e^{j\omega}) = \frac{A_0}{2} \left\{ W(e^{j(\omega-\omega_0)}) + W(e^{j(\omega+\omega_0)}) \right\} + \frac{A_1}{2} \left\{ W(e^{j(\omega-\omega_1)}) + W(e^{j(\omega+\omega_1)}) \right\}$$

copies of $W(e^{j\omega})$ at $\pm\omega_0, \pm\omega_1$
Two effects of the window

Resolution
- controlled by mainlobe width
- high resolution requires narrow ML

What controls mainlobe width?
- window length = primary factor
- window type = secondary factor
e.g., Hanning has wider ML than boxcar

Leakage
- controlled by sidelobe height
- energy at \( \omega_0 \) “leaks out” and can mask lower power signal at \( \omega_1 \)

What controls sidelobe height?
- type of window
- see table in Ch. 7

⇒ Consider example
Example (compare to 10.3 in text)

- 64-point rectangular window
- Mainlobe width $\approx \frac{4\pi}{64}$

\[
v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n)
\]

- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{6}$, $\omega_1 = \frac{2\pi}{3}$

$\Rightarrow$ copies of $W(e^{j\omega})$ at $\pm \frac{2\pi}{6}$, $\pm \frac{2\pi}{3}$
Example 10.3 continued

\[ |V(e^{j\omega})| \]

- \( A_0 = 1, A_1 = .75 \)
- \( \omega_0 = \frac{2\pi}{14}, \omega_1 = \frac{4\pi}{15} \)

- \( A_0 = 1, A_1 = .75 \)
- \( \omega_0 = \frac{2\pi}{14}, \omega_1 = \frac{2\pi}{12} \)

As sinusoids move closer together, leakage causes change in amplitudes
Example 10.3 continued

- \( A_0 = 1, \ A_1 = .75 \)
- \( \omega_0 = \frac{2\pi}{14}, \ \omega_1 = \frac{4\pi}{25} \)

- Sinusoids not resolved when they move close enough

- Mainlobe width
  \[ \approx \frac{4\pi}{64} = 0.0625\pi \]

- \( \omega_1 - \omega_0 \approx 0.0171\pi \)

⇒ Consider Matlab demo
Matlab demo: windowsine_demol

Parameters:

- Sampling frequency: $f_s = 100$ Hz
- Window length: 100 points $\rightarrow \approx 1$ second of data
- DFT length: 8192 points (lots of samples!)
- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a sinewave of varying frequency: 10 Hz (1st case) to 5 Hz (last case)

$\Rightarrow$ See what happens to spectral plot
Spectral sampling

Consider a 64-point section of 
\[ v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n) \]

- \( A_0 = 1, \ A_1 = .75 \)
- \( \omega_0 = \frac{2\pi}{14}, \ \omega_1 = \frac{4\pi}{25} \)

\[ |V[k]| \]

\[ |V(e^{j\omega})| \]

- Results of 64-point DFT
- DTFT (shown between 0 and \( 2\pi \))

DFT is samples of the DTFT ⇒
Spectral sampling can be misleading!

Consider a 64-point section of $v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n)$

- $A_0 = 1$, $A_1 = .75$, $\omega_0 = \frac{2\pi}{16}$, $\omega_1 = \frac{2\pi}{8}$

- Results of 64-point DFT
- Samples at zero-crossings

We can zero-pad $v[n]$ and take a larger FFT to get more samples ⇒
Zero-padding

Results of padding $v[n]$ with 64 zeros and taking 128-pt DFT

- Samples no longer at zero-crossings of DTFT
- "Clean" look of 64-point DFT was a bit of an illusion because samples occurred at zero-crossings of spectrum

Can zero-padding improve resolution?

Consider Matlab demo ⇒
Zero-padding Matlab demo: windowsine_demo2

Parameters:

- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a 5.5 Hz sinewave
- Sampling frequency: $f_s = 100$ Hz
- Window length: 100 points $\rightarrow \approx 1$ second of data
- DFT length varies from 128 to 32768

See what happens to spectral plot →
Why aren’t the sinusoids resolved in `windowsine_demo2`?

Window: 100 point rectangular window

Mainlobe width:  
\[ \Delta \omega_{\text{ML}} = \frac{4\pi}{100} \quad \rightarrow \quad \Delta \Omega_{\text{ML}} = \frac{4\pi}{100} f_s = \frac{4\pi(100)}{100} = 4\pi \]

Thus the mainlobe width corresponds to approximately 2 Hz

Since the sinewaves are 0.5 Hz apart, they won’t be resolved

⇒ Better resolution requires longer window (more data!)

Consider third Matlab demo ⇒
Matlab demo: windowsine_demo3

This demo illustrates how increasing the window length improves resolution.

Parameters:

- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a 5.5 Hz sinewave
- Sampling frequency: $f_s = 100$ Hz
- DFT length: 8192 points (lots of samples!)
- Window length varies from 100 to 1000 in increments of 25

See what happens to spectral plot ⇒