Fourier Analysis of Signals Using the DFT

Overview:

- Motivation

  Many applications require analyzing the frequency content of signals

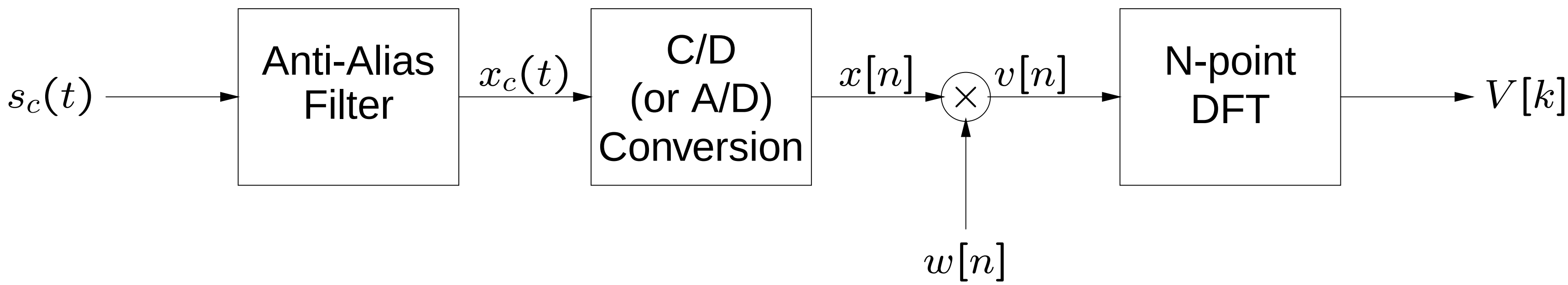
  - Speech processing – study resonances of vocal tract
  - Radar – Doppler processing
  - Sonar – study channel effects on long-range transmissions

- Basic spectral analyzer setup

  - Windowing
  - Frequency-sampling
  - Relating DFT output to CT spectrum

- Examples
- Time-dependent Fourier transform

Basic spectral analyzer



- Anti-alias filtering and A/D conversion
  - for tonight assume negligible errors
- Windowing operation
  - Necessary because DFT only works on finite-length sequences.
  - Typically use same windows as for filter design
- Frequency-sampling
  - Use FFT to compute samples of the DTFT

Understanding windowing & frequency-sampling is crucial! $\Rightarrow$

Windowing and frequency-sampling

What does the windowing operation do?

$$v[n] = w[n] \cdot x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} V(e^{j\omega}) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta}_{\text{convolution of } X(e^{j\omega}) \text{ with } W(e^{j\omega})}$$

- Convolution with window function smears the spectrum of $x[n]$
- Smoothes sharp peaks and discontinuities

Frequency-sampling

What is the output of the N-point DFT operation?

$$
\begin{aligned}
V[k] &= \sum_{n=0}^{N-1} v[n] e^{-j\frac{2\pi kn}{N}} \qquad\qquad k = 0, \ldots, N-1 \\
&= V(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}
\end{aligned}
$$

- DFT is samples of the DTFT evenly spaced around the unit circle

Relating DFT output to the CT spectrum

Suppose $x_c(t)$ is sampled at a rate of $f_s = \frac{1}{T}$.
What CT frequencies correspond to the N DFT samples?

Recall relationship between DT and CT frequencies: $\omega = \Omega T$

Thus the CT frequency associated with the kth bin is:

$$\Omega_k = \frac{\omega_k}{T} = \frac{2\pi k}{NT} = \frac{2\pi k f_s}{N}$$

Using this relationship, create a vector of analog frequencies to plot the spectrum against

Consider analyzing a signal consisting of 2 sinusoids $\Rightarrow$

Two sinusoid example

Consider $s_c(t)$ equal to the sum of 2 sinusoids: (sinusoid = sum of 2 eigenfxns)

$$s_c(t) = A_0 \cos(\Omega_0 t) + A_1 \cos(\Omega_1 t)$$

The sampled signal $x[n]$ becomes:

$$x[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n) \qquad \omega_0 = \Omega_0 T \qquad \omega_1 = \Omega_1 T$$

The windowed signal $v[n]$ is:

$$\begin{aligned} v[n] &= A_0 w[n] \cos(\omega_0 n) + A_1 w[n] \cos(\omega_1 n) \\ &= \frac{A_0}{2} w[n] \left(e^{j\omega_0 n} + e^{-j\omega_0 n} \right) + \frac{A_1}{2} w[n] \left(e^{j\omega_1 n} + e^{-j\omega_1 n} \right) \end{aligned}$$

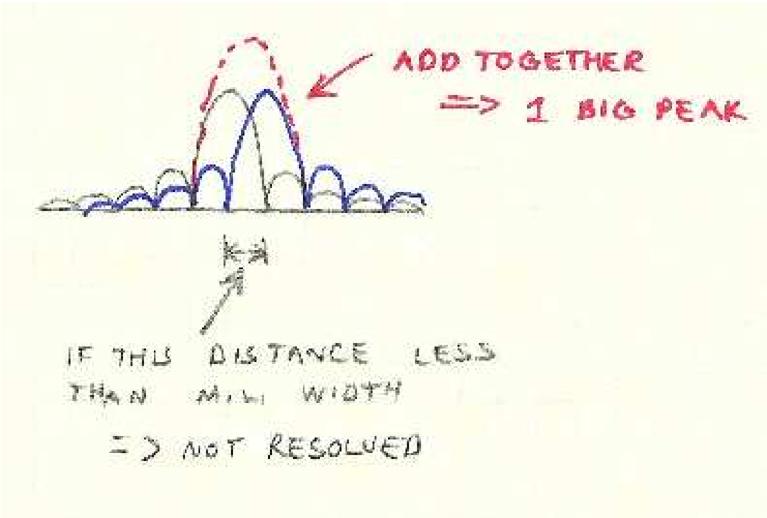
Taking the DTFT:

$$V(e^{j\omega}) = \underbrace{\frac{A_0}{2} \left\{ W(e^{j(\omega-\omega_0)}) + W(e^{j\omega+\omega_0}) \right\} + \frac{A_1}{2} \left\{ W(e^{j(\omega-\omega_1)}) + W(e^{j\omega+\omega_1}) \right\}}_{\text{copies of } W(e^{j\omega}) \text{ at } \pm\omega_0,\, \pm\omega_1}$$

Two effects of the window

<u>Resolution</u>

– controlled by mainlobe width
– high resolution requires narrow ML

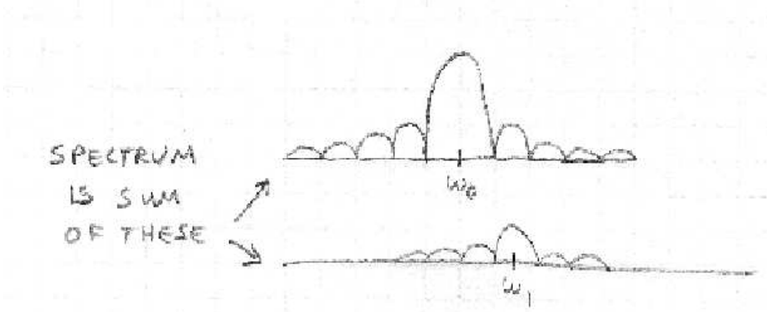


What controls mainlobe width?

– window length = primary factor
– window type = secondary factor
  e.g., Hanning has wider ML than boxcar

<u>Leakage</u>

– controlled by sidelobe height
– energy at ω_0 “leaks out” and can mask lower power signal at ω_1

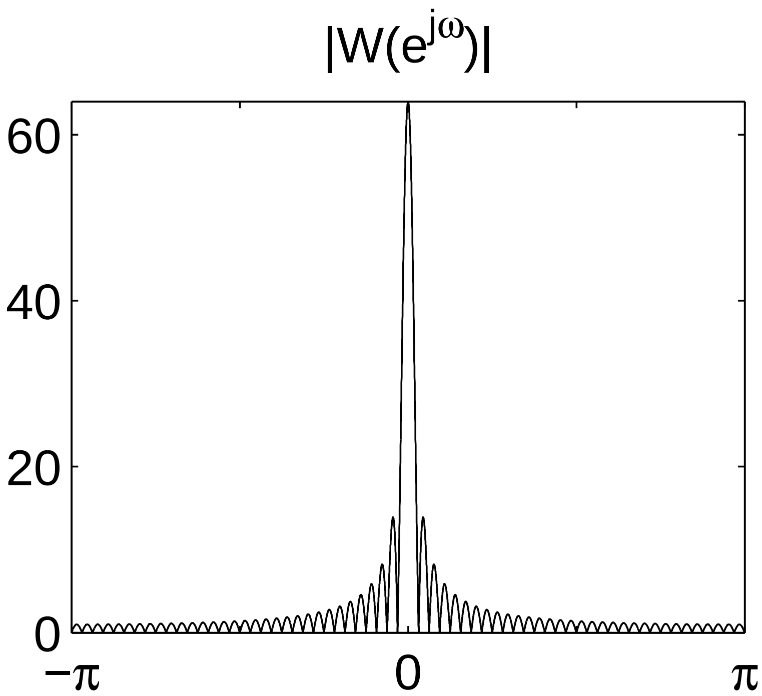


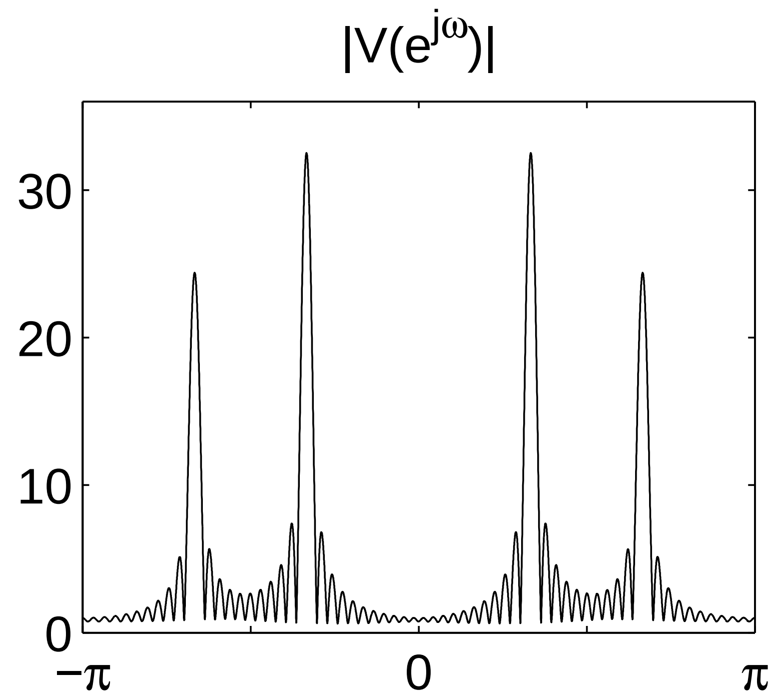
What controls sidelobe height?

– type of window
– see table in Ch. 7

$\Rightarrow$ Consider example

Example (compare to 10.3 in text)



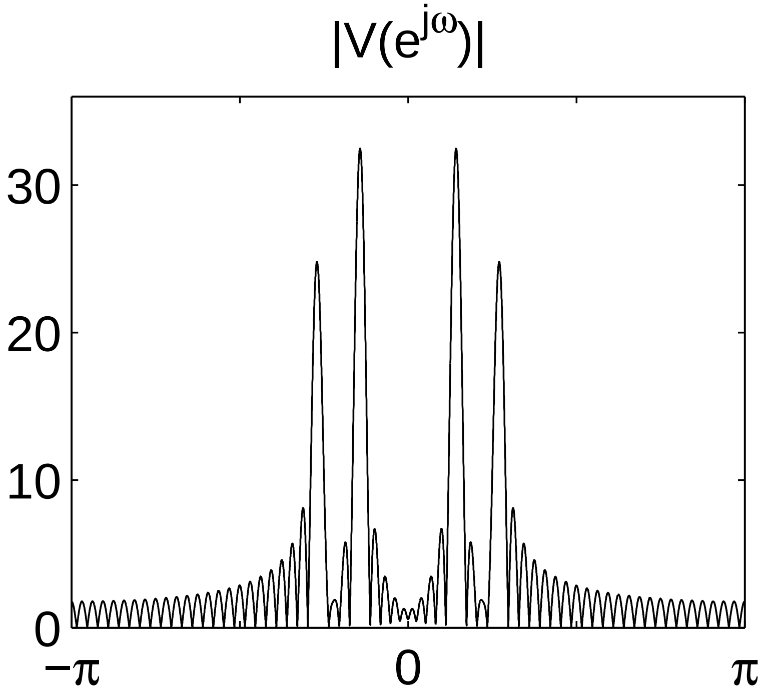


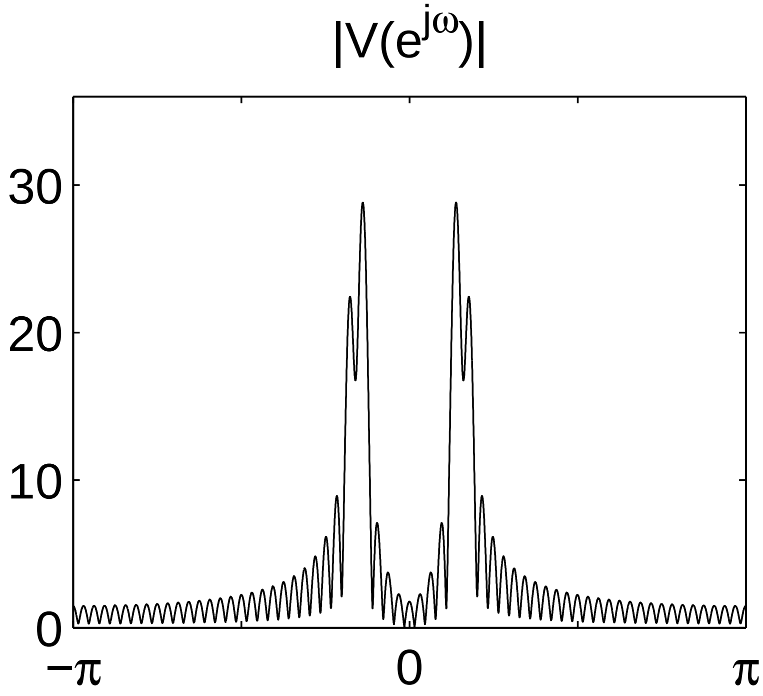
- 64-point rectangular window
- Mainlobe width $\approx \frac{4\pi}{64}$

- $v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n)$
- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{6}$, $\omega_1 = \frac{2\pi}{3}$

$\Rightarrow$ copies of $W(e^{j\omega})$ at $\pm\frac{2\pi}{6}$, $\pm\frac{2\pi}{3}$

Example 10.3 continued



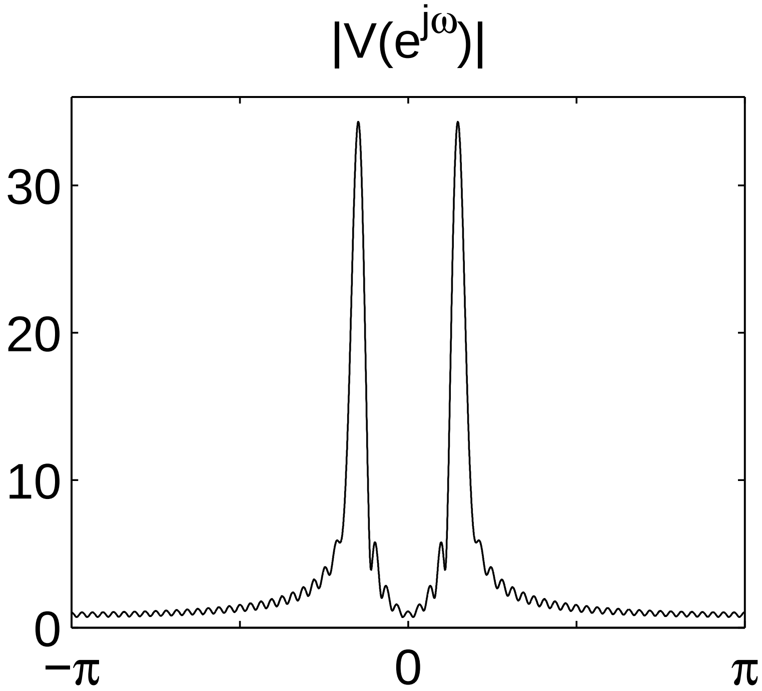


- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{14}, \omega_1 = \frac{4\pi}{15}$

- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{14}, \omega_1 = \frac{2\pi}{12}$

As sinusoids move closer together, leakage causes change in amplitudes

Example 10.3 continued



- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{14}$, $\omega_1 = \frac{4\pi}{25}$

- Sinusoids not resolved when they move close enough
- Mainlobe width $\approx \frac{4\pi}{64} = 0.0625\pi$
- $\omega_1 - \omega_0 = 0.0171\pi$

$\Rightarrow$ Consider Matlab demo

Matlab demo: `windowsine_demo1`

Parameters:

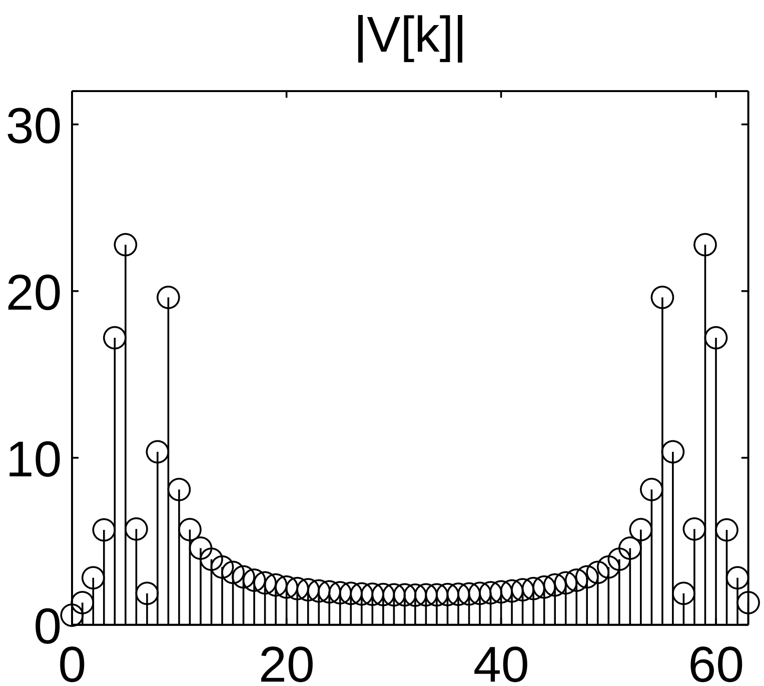
- Sampling frequency: $f_s = 100$ Hz
- Window length: 100 points $\quad \rightarrow \quad \approx$ 1 second of data
- DFT length: 8192 points (lots of samples!)
- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a sinewave of varying frequency: 10 Hz (1st case) to 5 Hz (last case)

$\Rightarrow$ See what happens to spectral plot

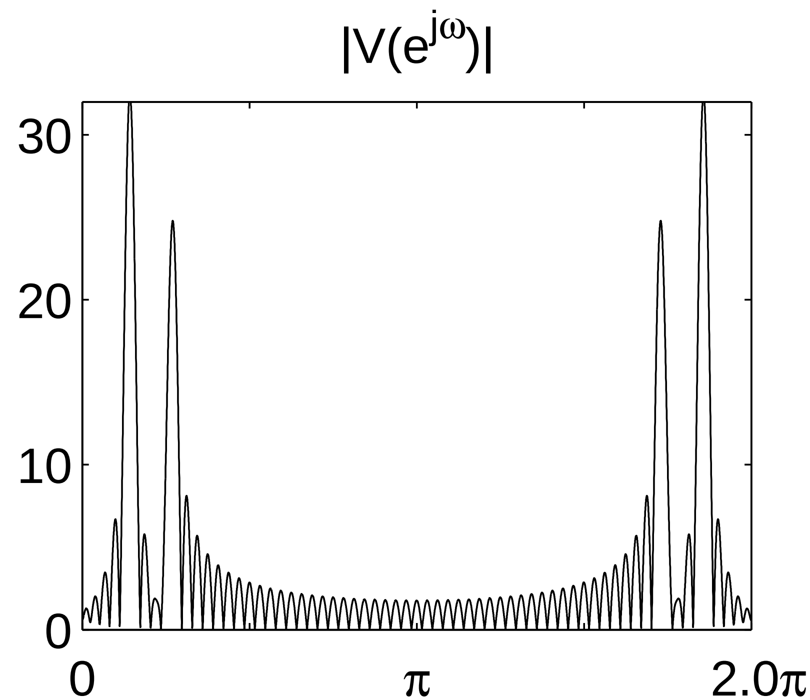
Spectral sampling

Consider a 64-point section of $v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n)$

- $A_0 = 1$, $A_1 = .75$
- $\omega_0 = \frac{2\pi}{14}$, $\omega_1 = \frac{4\pi}{25}$



– Results of 64-point DFT



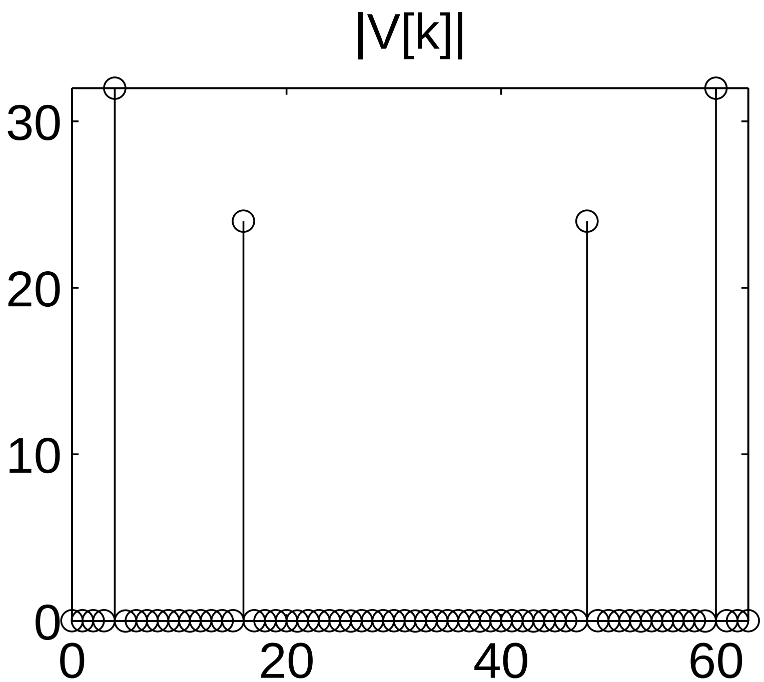
– DTFT (shown between 0 and 2π)

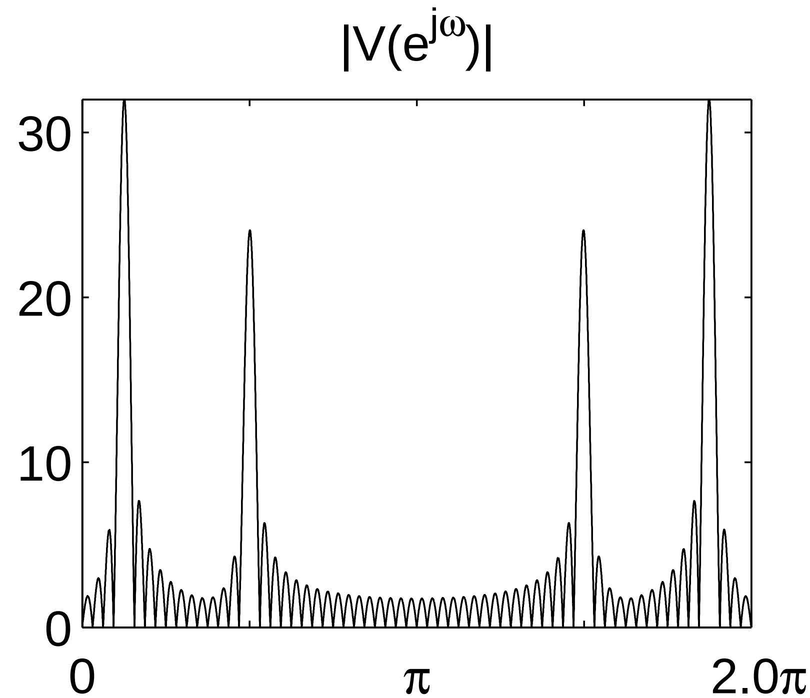
DFT is samples of the DTFT $\Rightarrow$

Spectral sampling can be misleading!

Consider a 64-point section of $v[n] = A_0 \cos(\omega_0 n) + A_1 \cos(\omega_1 n)$

– $A_0 = 1$, $A_1 = .75$, $\omega_0 = \frac{2\pi}{16}$, $\omega_1 = \frac{2\pi}{8}$





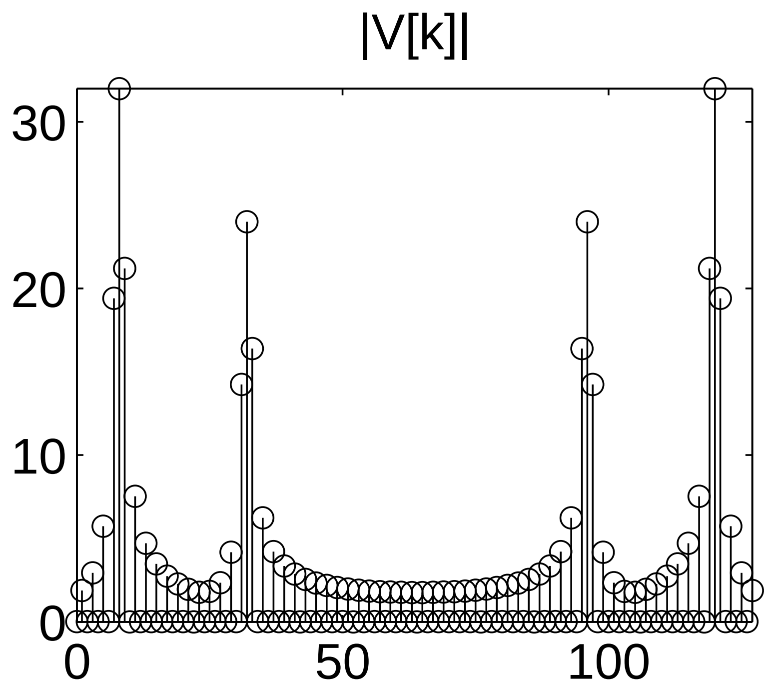
– Results of 64-point DFT

– Samples at zero-crossings

– DTFT (shown between 0 and 2π)

We can zero-pad $v[n]$ and take a larger FFT to get more samples $\Rightarrow$

Zero-padding



$\longleftarrow$ Results of padding $v[n]$ with 64 zeros and taking 128-pt DFT

- Samples no longer at zero-crossings of DTFT
- "Clean" look of 64-point DFT was a bit of an illusion because samples occured at zero-crossings of spectrum

Can zero-padding improve resolution?

Consider Matlab demo $\Rightarrow$

Zero-padding Matlab demo: `windowsine_demo2`

Parameters:

- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a 5.5 Hz sinewave
- Sampling frequency: $f_s = 100$ Hz
- Window length: 100 points $\quad \rightarrow \quad \approx$ 1 second of data
- DFT length varies from 128 to 32768

See what happens to spectral plot $\Rightarrow$

Why aren't the sinusoids resolved in `windowsine_demo2`?

Window: 100 point rectangular window

Mainlobe width: $\Delta\omega_{\mathrm{ML}} = \frac{4\pi}{100} \longrightarrow \Delta\Omega_{\mathrm{ML}} = \frac{4\pi}{100} f_s = \frac{4\pi(100)}{100} = 4\pi$

Thus the mainlobe width corresponds to approximately 2 Hz

Since the sinewaves are 0.5 Hz apart, they won't be resolved

$\Rightarrow$ Better resolution requires longer window (more data!)

Consider third Matlab demo $\Rightarrow$

Matlab demo: `windowsine_demo3`

This demo illustrates how increasing the window length improves resolution

Parameters:

- $x_c(t)$ consists of the sum of two sinewaves
  - a 5 Hz sinewave
  - a 5.5 Hz sinewave
- Sampling frequency: $f_s = 100$ Hz
- DFT length: 8192 points (lots of samples!)
- Window length varies from 100 to 1000 in increments of 25

See what happens to spectral plot $\Rightarrow$