ECE 754

* TURN IN PS 1
* PICK UP 1 HANDOUT
* DO IN-CLASS PROBLEM (NOT A QUIZ — CAN USE NOTES, ETC.)

IN-CLASS PROBLEM

SUPPOSE THAT YOU ARE WORKING WITH A LINEAR SONAR ARRAY. IT HAS 25 SENSORS WITH 2 METER SPACING. YOU WANT TO RECEIVE NARROWBAND SIGNALS WITH CENTER FREQUENCY OF 75 Hz. THE SPEED OF SOUND UNDERWATER IS 1500 M/S.

a) ASSUMING A UNIFORMLY-WEIGHTED ARRAY, WHAT IS THE MAINLOBE WIDTH OF THE FREQ-WAVENUMBER RESPONSE? GIVE YOUR ANSWER IN TERMS OF k_z.

M.L. WIDTH = NULL-NULL $\dfrac{4\pi}{dN} = \dfrac{4\pi}{2 \cdot 25} = \dfrac{4\pi}{50}$

NO ALIASING: $d \leq \dfrac{\lambda}{2} = \dfrac{c/f}{2} = \dfrac{20m}{2}$

$d \leq 10m$

b) SUPPOSE YOU WANT TO INCREASE THE RESOLUTION OF THE ARRAY. WHICH OF THE FOLLOWING WOULD YOU DO:

i) ADD SENSORS, KEEPING THE OVERALL LENGTH OF THE ARRAY (i.e, LENGTH OF CABLE SENSORS ARE ON) THE SAME.

ii) KEEP SAME NUMBER OF SENSORS, BUT SPREAD THEM OUT OVER A LONGER CABLE.

EXPLAIN, IS THERE A LIMIT TO # OF SENSORS AND/OR LENGTH OF CABLE THAT SHOULD BE USED?

LAST TIME: PLANEWAVE MODEL FOR SPATIAL SIGNALS

$$f(t, \underline{p}) = \iint F(\omega, \underline{k})\, e^{j(\omega t - \underline{k}^T \underline{p})}\, d\omega\, d\underline{k}$$

TIME LOCATION $\underline{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

MULTIDIMENSIONAL FOURIER SYNTHESIS

BASIS FUNCTIONS: PLANEWAVES

$$e^{j(\omega t - \underline{k}^T \underline{p})}$$

NARROWBAND

$\underline{k}$ = WAVENUMBER

$$= -\frac{2\pi}{\lambda}\underline{u} = \frac{-2\pi}{\lambda}\begin{bmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}$$

λ = WAVELENGTH

SPATIAL FILTERING:

$$e^{j(\omega t - \underline{k}^T \underline{p})} \longrightarrow \boxed{\text{NARROWBAND SPATIAL FILTER}} \longrightarrow \Upsilon(\omega, \underline{k})\, e^{j(\omega t - \underline{k}^T \underline{p})}$$

FREQ - WAVENUMBER RESPONSE

FREQ - WAVENUMBER RESPONSE FOR SENSOR ARRAY:

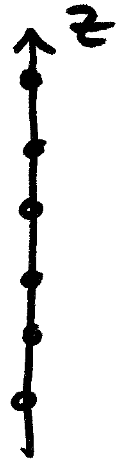
$$\Upsilon(\omega, \underline{k}) = \underline{w}^H \underline{v}_k(\underline{k})$$

$\underline{w}$ = WEIGHT VECTOR

$\underline{v}_k(\underline{k})$ = ARRAY MANIFOLD VECTOR

"REPLICA VECTOR"

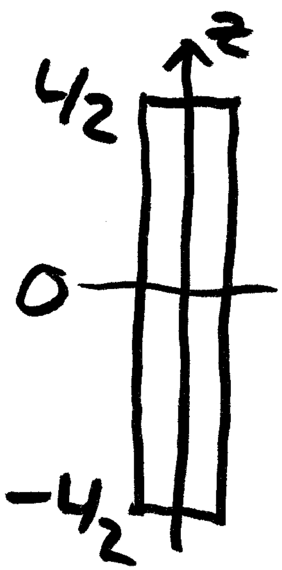
LAST TIME: LINEAR, EQUAL-SPACED ARRAY



$$p_{zn} = \left(n - \left(\frac{N-1}{2}\right) d\right)$$

$$\underline{I}(\omega, \underline{k}) = \frac{1}{N} \frac{\sin\left(\frac{N k_z d}{2}\right)}{\sin\left(\frac{k_z d}{2}\right)}$$

EXTEND RESULTS TO CONTINUOUS APERTURE:



MEASUREMENTS ON CONTINUUM

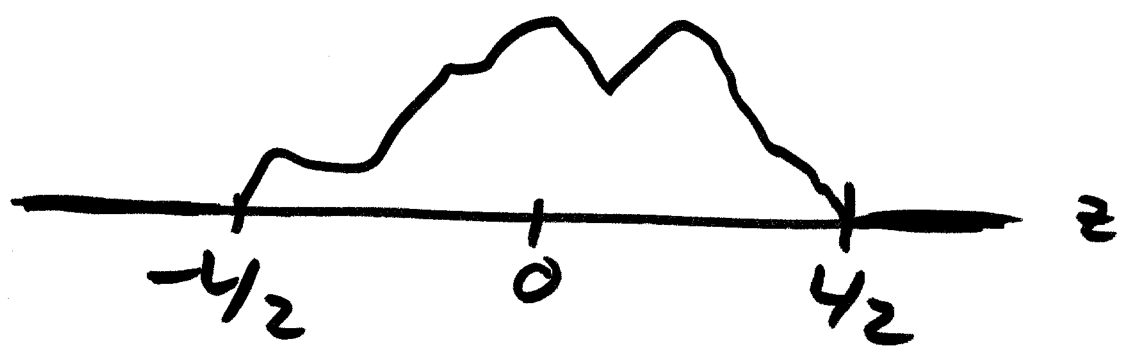
$$-\frac{L}{2} \leq z \leq \frac{L}{2}$$

MEASURE NB SIGNAL $F(\omega, z)$

FREQ - WAVE # RESP. FOR APERTURE

$$\mathcal{I}_a(\omega, k_z) = \int_z w_a^*(z)\, e^{-jk_z z}\, dz$$

$w_a(z)$ = CONTINUOUS APERTURE WEIGHTING FXN



$\mathcal{I}_a(\omega, k_z)$ = SPATIAL FOURIER TRANSFORM OF $w_a^*(z)$

SPATIAL FREQ. RESP. OF APERTURE

CONSIDER: UNIFORMLY WEIGHTED APERTURE

EX

$$w_a(z) = \begin{cases} \frac{1}{L} & |z| \le L/2 \\ 0 & \text{OTHERWISE} \end{cases}$$

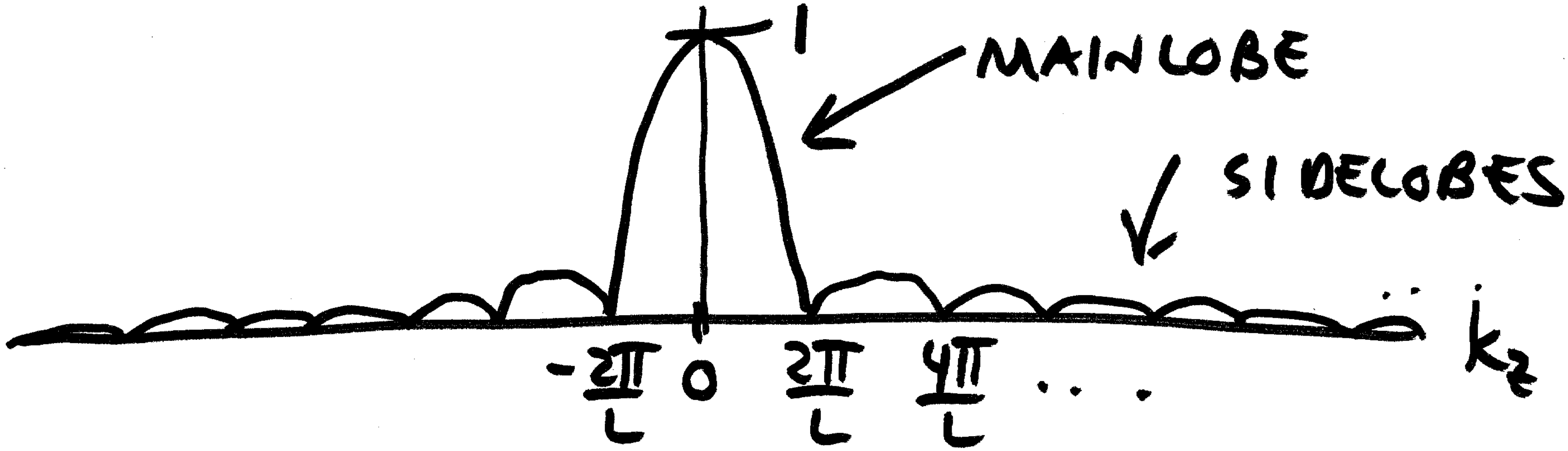
$$I_a(\omega, k_z) = \int_{-L/2}^{+L/2} \frac{1}{L} e^{-jk_z z}\, dz = \frac{1}{L} \cdot \frac{1}{-jk_z} e^{-jk_z z} \Big|_{-L/2}^{+L/2}$$

$$= \frac{-1}{jk_z L} \left[e^{-jk_z L/2} - e^{+jk_z L/2} \right]$$

$$= \frac{\sin\left(\frac{k_z L}{2}\right)}{\frac{k_z L}{2}} = \mathrm{sinc}\left(\frac{k_z L}{2}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

$$\left| I_a(\omega, k_z) \right|$$



1ST ZERO CROSSING

$$\frac{k_z L}{2} = \pi M$$

$$k_z = \frac{2\pi M}{L}$$

HEIGHT AT Ø

L'HÔPITAL'S

$$\left. \frac{\frac{d}{dk_z}\left(\sin\left(\frac{k_z L}{2} \right) \right)}{\frac{d}{dk_z}\left(\frac{k_z L}{2} \right)} \right|_{k_z = 0} = 1$$

$\boxed{?}$ WHAT DETERMINES RESOLUTION OF THIS APERTURE?

$$\text{ML WIDTH} = \frac{4\pi}{L}$$

AS $L \rightarrow$ INCREASES, RESOLUTION GETS BETTER

SEPARATING PLANEWAVES IN SPACE IS ANALOGOUS TO SEPARATING SINUOSOIDS IN TIME

TO SEPARATE CLOSE SINEWAVES, TAKE LONG TIME SAMPLES

$\boxed{?}$ IF WE INCREASE L, HOW DO SIDELOBES CHANGE?

WIDTH BETWEEN S.L. RIPPLES NARROWS

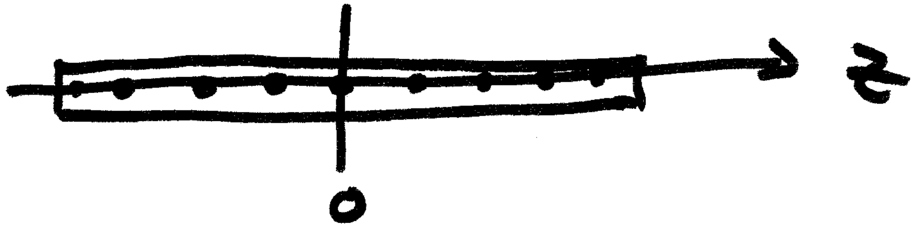
SPACE BETWEEN RIPPLES CHANGES

PEAK S.L. HEIGHT DOES NOT

ONLY WAY TO CHANGE S.L. STRUCTURE IS TO CHANGE $w_a(z)$

SMOOTHER

WE CAN VIEW AN ARRAY AS SAMPLED APERTURE"



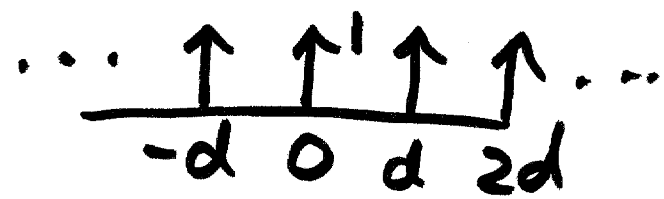
TAKE N SAMPLES (N ODD)

SAMPLE SPACING = d

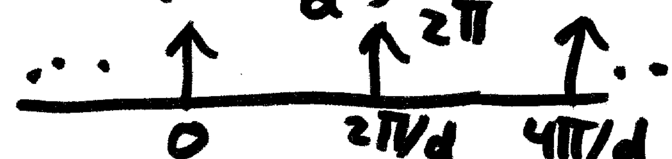
FROM PS 1: MODEL FOR SAMPLING

MULT. BY IMPULSE TRAIN

$$w[n] = w_a(z) \sum_{n=-\infty}^{+\infty} \delta(n - dz)$$



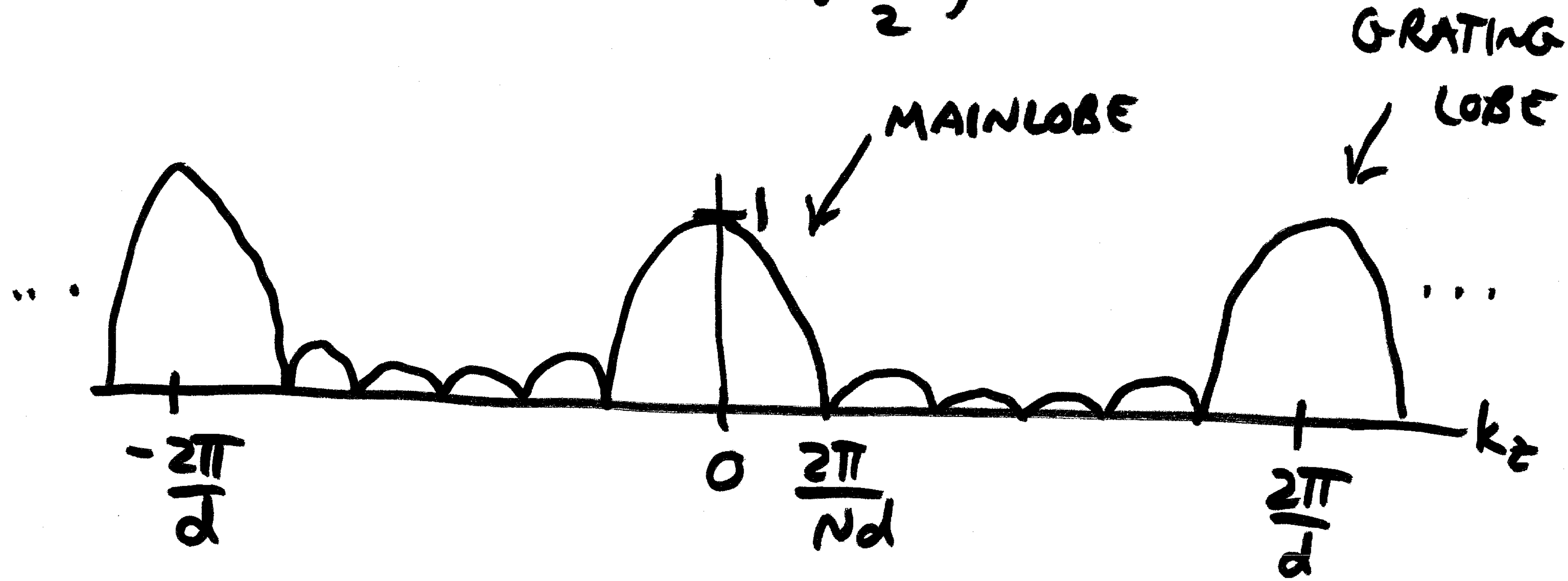
$$I(\omega, k_z) = \frac{1}{2\pi} I_a(\omega, k_z) * \sum_{n=-\infty}^{+\infty} 2\pi \delta\left(k_z - n\frac{2\pi}{d}\right)$$



$$I(\omega, k_z) = \sum_{n=-\infty}^{+\infty} I_a\left(\omega, k_z - \frac{2\pi}{d} n\right)$$

↑ COPY OF I_a EVERY $\frac{2\pi}{d}$

$$= \frac{1}{N} \frac{\sin\left(\frac{N k_z d}{2}\right)}{\sin\left(\frac{k_z d}{2}\right)}$$



SAMPLED APERTURE HAS PERIODIC STRUCTURE

GRATING LOBES: REGULAR INTERVALS

k_z-SPACE: EVERY $\frac{2\pi}{d}$ SPATIAL SAMPLE FREQ.

ψ-SPACE: EVERY 2π $\psi = -k_z d$

u_z-SPACE: EVERY $\frac{\lambda}{d}$ $u_z = \frac{-\lambda}{2\pi} k_z$

SPATIAL SPECTRUM IS BANDLIMITED:

VISIBLE REGION: $-\frac{2\pi}{\lambda} \leq k_z \leq \frac{2\pi}{\lambda}$

B.L. TO $\frac{2\pi}{\lambda}$

NYQUIST: SAMPLE FREQ > 2 (HIGHEST)

$$\frac{\cancel{2\pi}}{d} > 2\left(\frac{\cancel{2\pi}}{\lambda}\right)$$

$$d < \frac{\lambda}{2}$$

TO HAVE AN APERTURE W/ SAME M.L.
WIDTH AS A LINEAR ARRAY

$$\frac{2\pi}{L} = \frac{2\pi}{Nd}$$

$$L = Nd$$

L = APERTURE LENGTH

$(N-1)d$ = LENGTH OF ARRAY

AS $d \longrightarrow 0$: ARRAY BECOMES AN APERTURE

SUMMARIZE: SAMPLING ANALYSIS

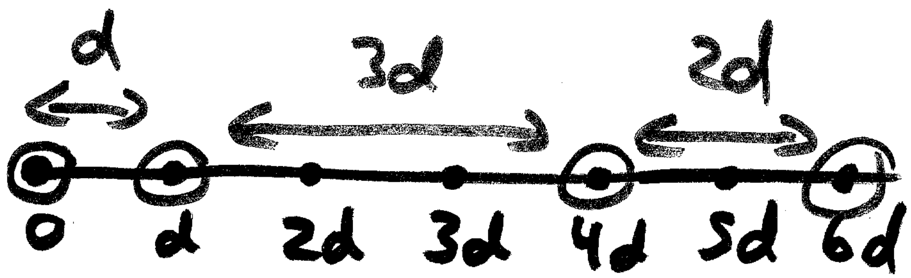
PERIODIC SAMPLING RESULTS IN

- GRATING LOBES AT $\frac{2\pi}{d}$ IN k_z
- NO ALIASING IF $d < \frac{\lambda}{2}$
- RESOLUTION: ML WIDTH INVERSELY PROPORTIONAL TO LENGTH

NON-UNIFORM SPACING

IF THERE IS AN UNDERLYING GRID THEN, WE'LL GET GRATING LOBES AT $\frac{2\pi}{d_{GRID}}$

CO-ARRAY



AVOID GRATING LOBES + STILL HAVE LONG APERTURE W/ NON-UNIFORM SPACING

ARRAY STEERING

TWO OPTIONS:

① MECHANICAL STEERING

ROTATE ARRAY SO DESIRED SIGNAL IS AT $\theta = \pi/2$

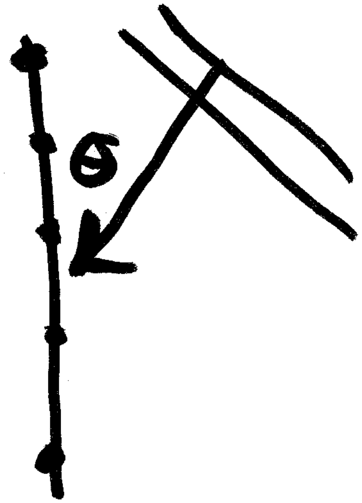
ELECTRICAL STEERING

CHANGE $\underline{w}$ SO THAT M.L. OF $I(\omega, \underline{k})$ SHIFTS TO $\underline{k}_T$ = DESIRED WAVE #

$\boxed{?}$ HOW TO CHANGE WEIGHTS

$$\underline{w} = \underline{v}_k(\underline{k}_T)$$

SPATIAL MATCHED FILTER



TIME ALIGN + THEN ADD UP

TIME SHIFT + ADD

NB SIGNAL : TIME-SHIFT → PHASE SHIFT

$e^{j\omega_0 t}$

CALCULATED PHASE SHIFTS LAST TIME

$e^{+j\underline{k}^T \underline{\rho}}$

$\Rightarrow \quad \underline{w} = \frac{1}{N} \underline{v}_K(\underline{k}_T) \qquad \underline{k}_T =$ STEERING DIRECTION

RESULTING FREQ - WAVE # RESP.

$$I_{STEER}(\omega, \underline{k}) = I(\omega, \underline{k} \mid \underline{k}_T) = \underline{w}^H \underline{v}_K(\underline{k})$$

$$= \underline{\underline{\frac{1}{N}}} \, \underline{v}_K^H(\underline{k}_T)\, \underline{v}_K(\underline{k})$$

CONVENTIONAL BEAMPATTERN

$$= \underline{I}(\omega, \underline{k} - \underline{k}_T)$$

↑ SIMPLE SHIFT OF ORIGINAL FREQ - WAVE # FXN

FOR A LINEAR ARRAY:

$$B_{STEER}(k_z \mid k_T) = B(k_z - k_T)$$

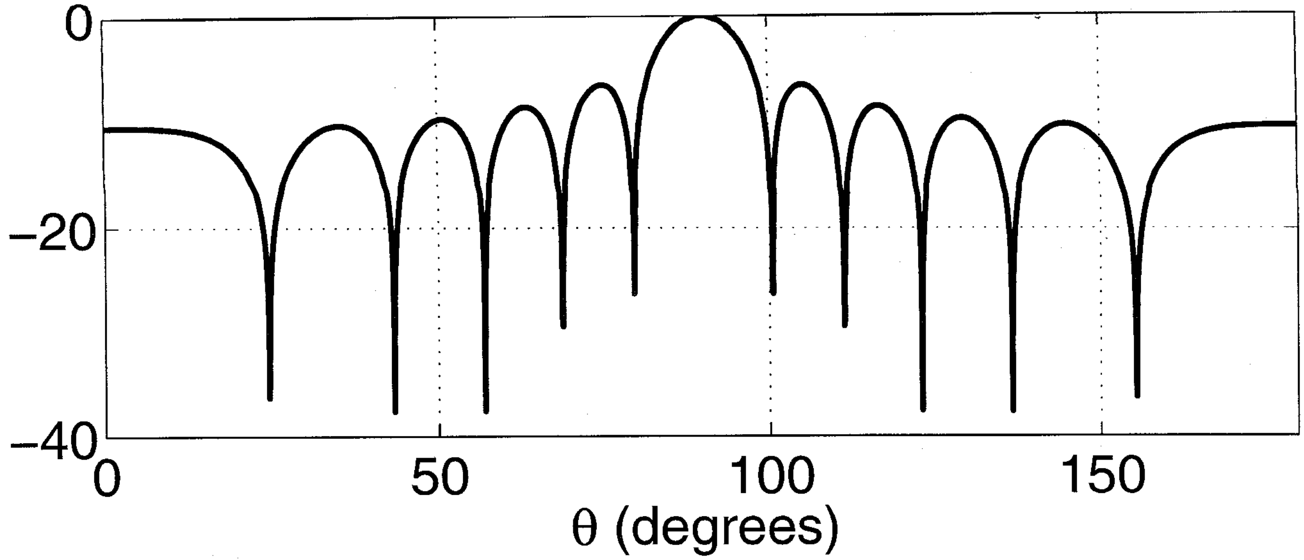
$$B_{STEER}(\psi \mid \psi_T) = B(\psi - \psi_T)$$

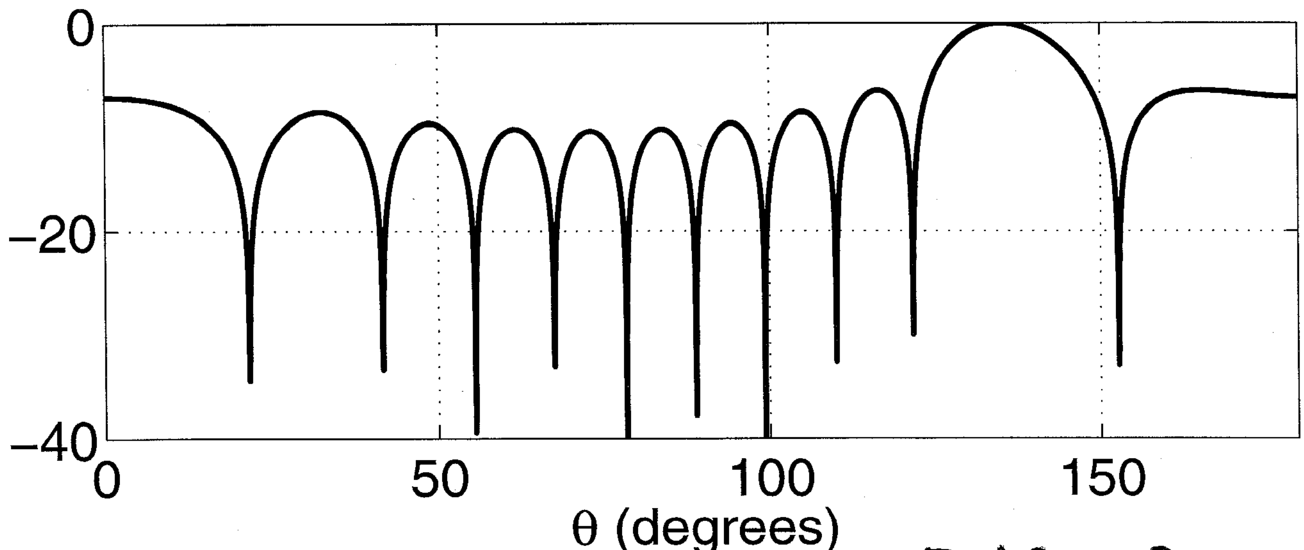
$$B_{STEER}(u_z \mid u_T) = B(u_z - u_T)$$

SIMPLE SHIFTS

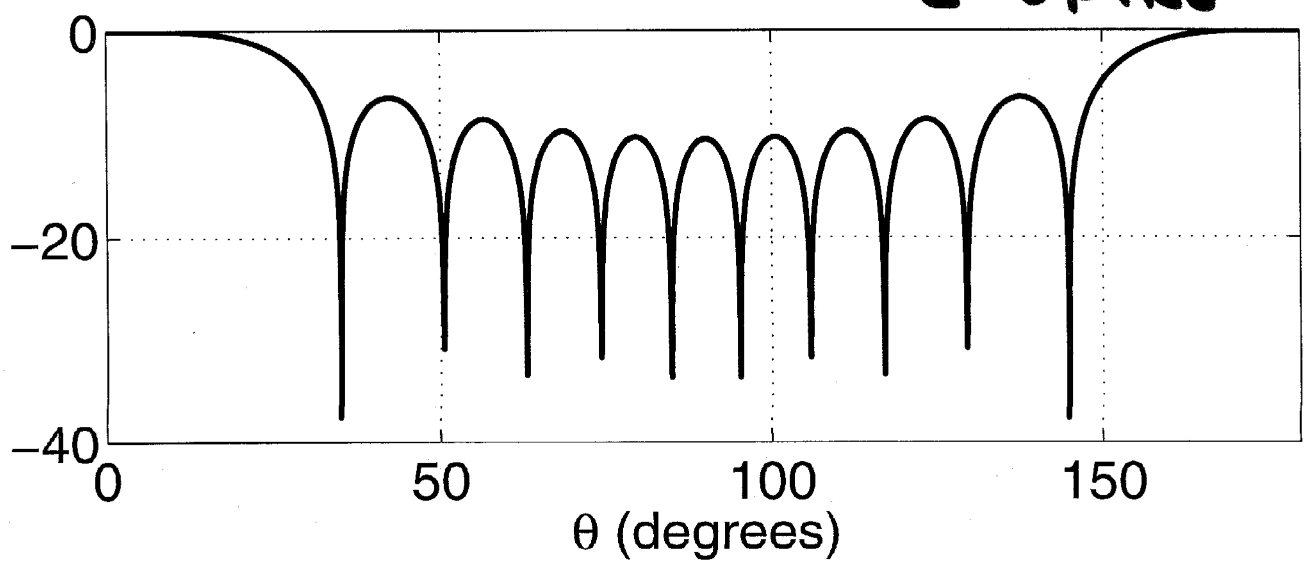
NOT TRUE IN θ!

BROADSIDE





ENDFIRE

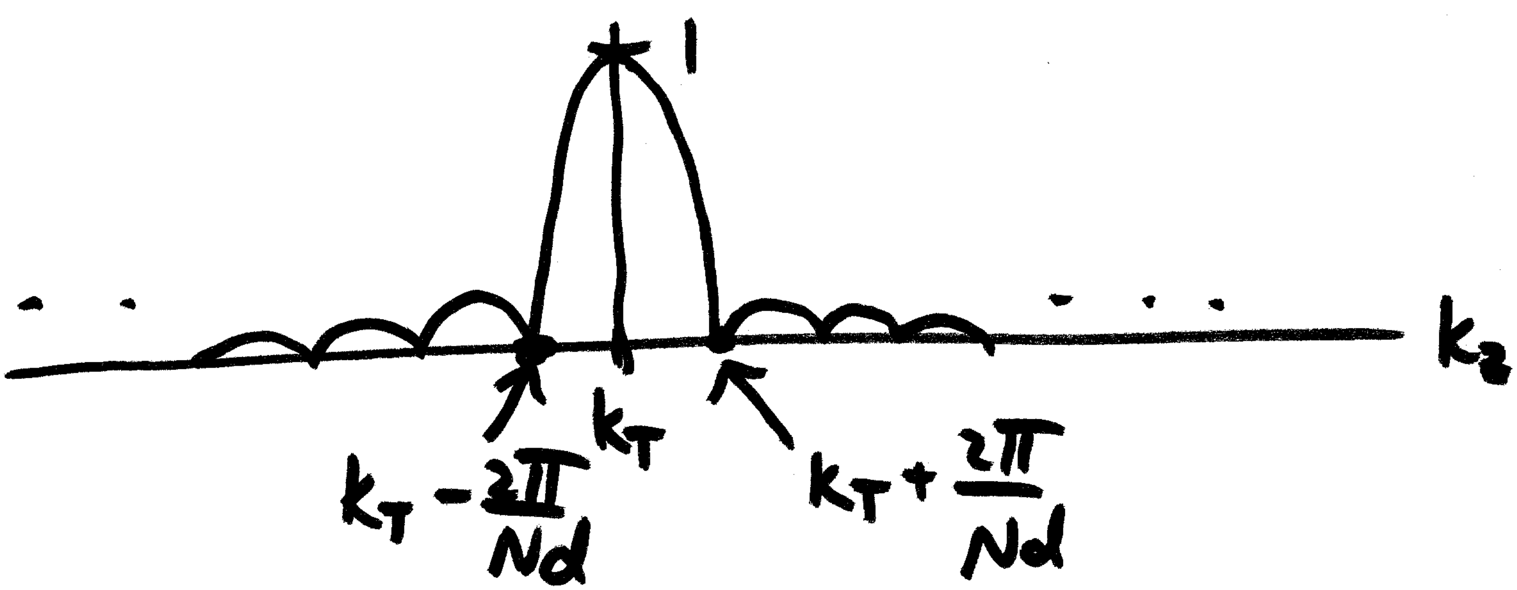


NONLINEAR SCALING:

$$k_z = -\frac{2\pi}{\lambda}\cos\theta$$

$$\theta = \operatorname{acos}\left(-\frac{\lambda}{2\pi}k_z\right)$$

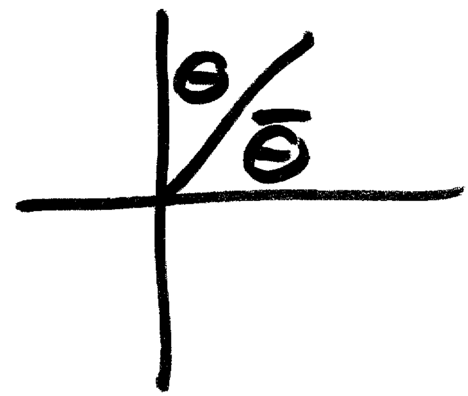
PLOT IN k_z, STEERED PATTERN:



$$k_{NULL} = k_T \pm \frac{2\pi}{Nd}$$

$$-\frac{2\pi}{\lambda}\cos(\theta_{NULL}) = -\frac{2\pi}{\lambda}\cos(\theta_T) \pm \frac{2\pi}{Nd}$$

$$\bar{\theta} = \frac{\pi}{2} - \theta$$



$$\cos(\theta) = \sin(\bar{\theta})$$

$$-\frac{2\pi}{\lambda}\sin(\bar{\theta}_{NULL}) = -\frac{2\pi}{\lambda}\sin(\bar{\theta}_T) \pm \frac{2\pi}{Nd}$$

$$\bar{\theta}_{NULL} = \bar{\theta}_T \pm \Delta\theta$$

$$\sin(\bar{\theta}_T \pm \Delta\theta) = \sin(\bar{\theta}_T) \pm \frac{\lambda}{Nd}$$

FIND FORMULA FOR $\pm\Delta\theta = \bar{\theta}_T - \text{asin}\left(\sin(\bar{\theta}_T) \pm \frac{\lambda}{Nd}\right)$

2 CASES: BROADSIDE

$$\bar{\theta}_T = 0$$

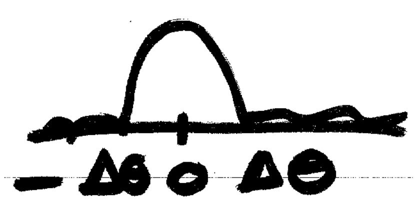
$$\sin(0 \pm \Delta\theta) = \sin(0) \pm \frac{\lambda}{Nd}$$

$$\sin(\pm\Delta\theta) = \pm\frac{\lambda}{Nd}$$

$$\Delta\theta = \text{asin}\left(\frac{\lambda}{Nd}\right)$$

SMALL ANGLE APPROX.

$$\Delta\theta \approx \frac{\lambda}{Nd}$$



CASE 2 : ENDFIRE $\bar{\theta}_T = \pi/2$

$$\sin\left(\frac{\pi}{2} \pm \Delta\theta\right) = \sin\left(\frac{\pi}{2}\right) \pm \frac{\lambda}{Nd}$$

$$\sin\left(\frac{\pi}{2}\right)\cos(\Delta\theta) \pm \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\sin(\Delta\theta) = \sin\left(\frac{\pi}{2}\right) \pm \frac{\lambda}{Nd}$$

$$\cos(\Delta\theta) = 1 \pm \frac{\lambda}{Nd}$$

$$1-\cos(\Delta\theta) = \mp \frac{\lambda}{Nd}$$

TRIG: $1-\cos(\alpha) = 2\sin^2\left(\frac{\alpha}{2}\right)$

$$2\sin^2\left(\frac{\Delta\theta}{2}\right) = \mp \frac{\lambda}{Nd}$$

$$\sin\left(\frac{\Delta\theta}{2}\right) = \sqrt{\frac{\lambda}{2Nd}}$$

SMALL ANGLE: $\sin(x) \approx x$

$$\frac{\Delta\theta}{2} \approx \sqrt{\frac{\lambda}{2Nd}}$$

$$\Delta\theta \approx \sqrt{\frac{2\lambda}{Nd}}$$

SUMMARY: BROADSIDE: $\Delta\theta \approx \dfrac{\lambda}{Nd}$

ENDFIRE: $\Delta\theta \approx \sqrt{\dfrac{2\lambda}{Nd}}$

BETTER RESOLUTION AT BROADSIDE

DIRECTIVITY

$$P(\theta, \phi) = \underset{\text{PATTERN}}{\text{POWER}} = |B(\omega, \theta, \phi)|^2$$

$$D = \frac{P(\theta_T, \phi_T)}{\frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta \, P(\theta, \phi)}$$

NORMALIZE SO $P(\theta_T, \phi_T) = 1$

STANDARD LINEAR ARRAY: EQUAL-SPACING $d = \frac{\lambda}{2}$

$$D = \frac{1}{\|\underline{w}^H \underline{w}\|^2}$$

MAXIMIZED FOR UNIFORM WEIGHTING

$$\mathcal{E}\{\underline{n}\} = 0$$

$$\mathcal{E}\{\underline{n}\,\underline{n}^H\} = \sigma_n^2 I$$

INDEPENDENT FROM SENSOR TO SENSOR.

NOISE POWER AT OUTPUT:

$$\mathcal{E}\{\underline{w}^H \underline{n}\,\underline{n}\,\underline{w}^H\} = \underline{w}^H \mathcal{E}\{\underline{n}\,\underline{n}^H\}\,\underline{w}$$

$$= \underline{w}^H \sigma_n^2 I\,\underline{w}$$

$$= \sigma_n^2\,\underline{w}^H \underline{w}$$

SNR GAIN:

$$\frac{\text{SNR - OUTPUT}}{\text{SNR - INPUT (SINGLE SENSOR)}} = \frac{\dfrac{1}{\sigma_n^2 \underline{w}^H \underline{w}}}{\dfrac{1}{\sigma_n^2}}$$

ARRAY GAIN VS. WHITE NOISE

(NOISE INDEPENDENT FROM SENSOR TO SENSOR)

UNITY GAIN CONSTRAINT IN LOOK DIRECTION: $\underline{w}^H \underline{v}_K(k_T) = 1$

INPUT AT A SENSOR: $f_n(t) = e^{j\omega t} e^{-j\underline{k}^T \underline{p}} + \text{noise}$

NOISE: σ_w^2 POWER

SIGNAL AT OUTPUT: $\underline{w}^H \underline{v}_K(k_T) = 1$

NOISE AT OUTPUT:

$= \underline{w}^H \underline{n}$

$$\text{GAIN} = A_w = \frac{1}{\underline{w}^H \underline{w}}$$

UNIFORM WEIGHTING: N

$$A_w \leq N$$

WHITE NOISE GAIN ALWAYS $\leq N$

$$T_{SE} = \underset{\text{FXN}}{\text{SENSITIVITY}} = A_w^{-1}$$