ECE 754

WARM-UP PROBLEM

GIVEN:
$$w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{OTHER} \end{cases}$$

FIND DTFT OF $w[n]$ + SKETCH $|W(e^{j\omega})|$

FIND z-TRANS. OF $w[n]$

SKETCH POLES + ZEROS OF $W(z)$

WARM-UP

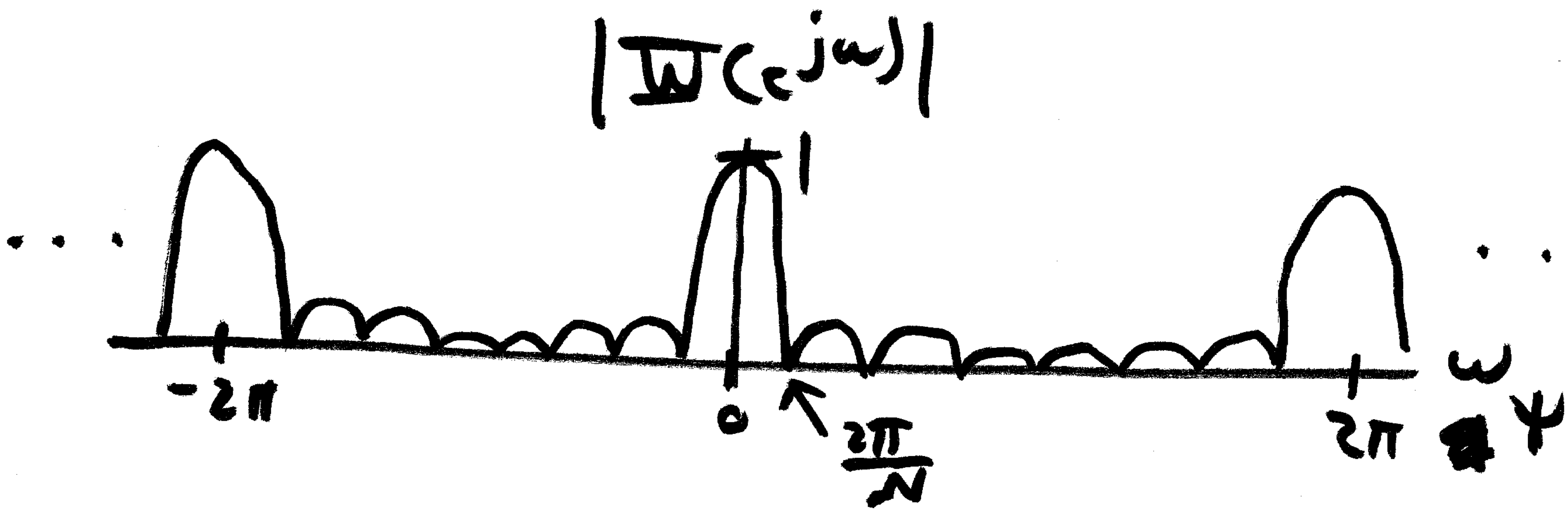
$$W(z) = \sum_n w[n] z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

$$W(e^{jw}) = \sum_n w[n] e^{-jwn} = \frac{1 - e^{-jwN}}{1 - e^{-jw}}$$

$\boxed{?}$ HOW DO z + FOURIER RELATE?

$$W(e^{jw}) = W(z)\Big|_{z = e^{jw}}$$

z ON UNIT CIRCLE



WARM-UP

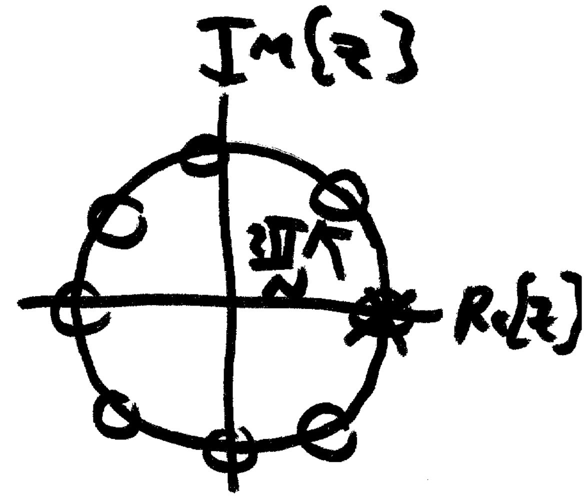
$$\underline{W}(z) = \frac{1 - z^{-N}}{1 - z^{-1}} \cdot \frac{z^N}{z^N}$$

POLES + ZEROS?

$$= \frac{z^N - 1}{z^{N-1}(z-1)}$$

ZEROS: $z^N = 1 = e^{j2\pi n}$

$$z = e^{j\frac{2\pi n}{N}}$$



POLES: $z^{N-1} = 0$

$z = 0$ N-1 POLES AT $z = 0$

1 POLE AT $z = 1$

CH. 3 : DESIGN PROBLEM

CHOOSE $\underline{w}$ TO MEET CERTAIN CRITERIA

EX) MINIMIZE BEAMWIDTH
(FILTER PASSBAND WIDTH)

CONTROL SIDELOBE LEVELS

UNCERTAINTY PRINCIPLE:

$$\sqrt{\overline{\Delta \ell^2}}\,\sqrt{\overline{\Delta k^2}} \geq \frac{1}{2}$$

NORMALIZED MEAN SQ. WIDTH OF APERTURE

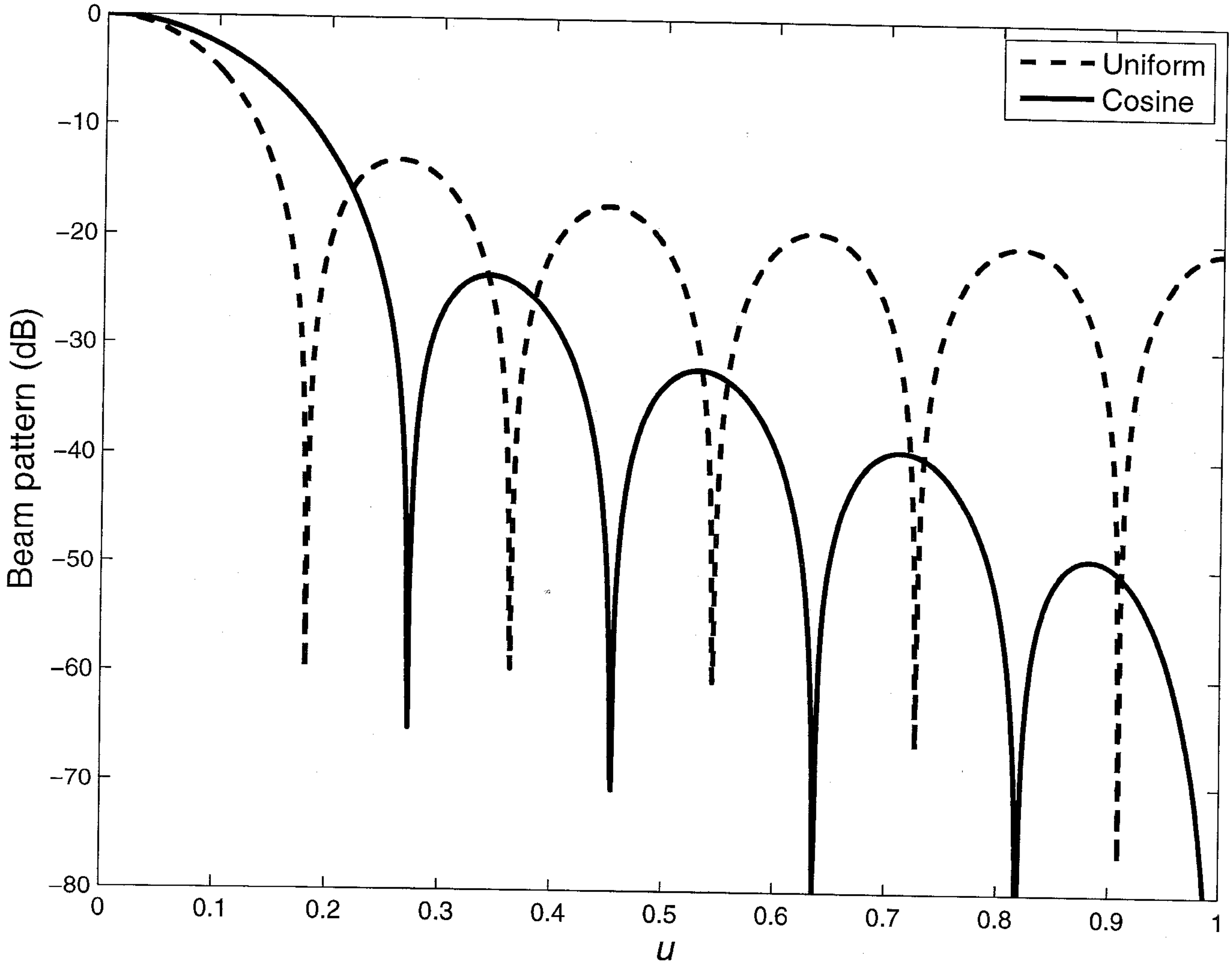
NORM. MEAN SQ WIDTH OF SPATIAL RESPONSE

FOCUS ON LINEAR, EQUALLY SPACED ARRAYS

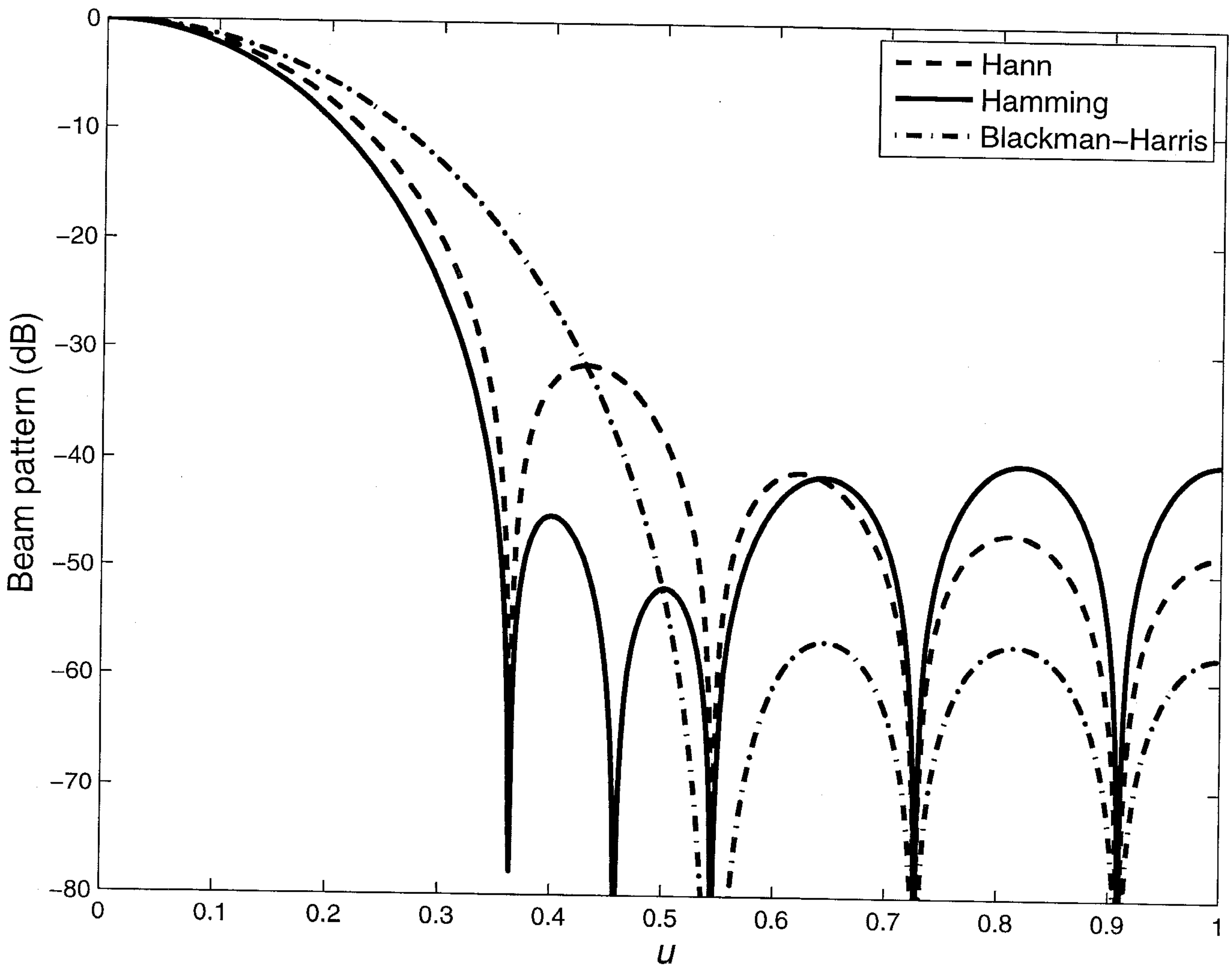
DESIGN METHODS:

- $\underline{w}$ = STANDARD WINDOW AS USED IN SPECTRAL ESTIMATION
- FREQUENCY SAMPLING
- MINIMUM BW FOR FIXED SIDELOBE LEVEL → DOLPH-CHEBYCHEV
- LEAST SQUARES
  MINIMIZE MSQ ERROR BETWEEN DESIRED + ACTUAL BEAM PATTERN
- MIN-MAX (PARKS-McCLELLAN)

3.1 OF TEXT



TRADE OFF: M.L. WIDTH
SIDELOBE HEIGHT



Hann
Hamming
Blackman–Harris
Beam pattern (dB)
0
−10
−20
−30
−40
−50
−60
−70
−80
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
u

On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform

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Abstract–This paper makes available a concise review of data windows and their affect on the detection of harmonic signals in the presence of broad-band noise, and in the presence of nearby strong harmonic interference. We also call attention to a number of common errors in the application of windows when used with the fast Fourier transform. This paper includes a comprehensive catalog of data windows along with their significant performance parameters from which the different windows can be compared. Finally, an example demonstrates the use and value of windows to resolve closely spaced harmonic signals characterized by large differences in amplitude.

I. Introduction

THERE IS MUCH signal processing devoted to detection and estimation. Detection is the task of determining if a specific signal set is present in an observation, while estimation is the task of obtaining the values of the parameters describing the signal. Often the signal is complicated or is corrupted by interfering signals or noise. To facilitate the detection and estimation of signal sets, the observation is decomposed by a basis set which spans the signal space [1]. For many problems of engineering interest, the class of signals being sought are periodic which leads quite naturally to a decomposition by a basis consisting of simple periodic functions, the sines and cosines. The classic Fourier transform is the mechanism by which we are able to perform this decomposition.

By necessity, every observed signal we process must be of finite extent. The extent may be adjustable and selectable, but it must be finite. Processing a finite-duration observation imposes interesting and interacting considerations on the harmonic analysis. These considerations include detectability of tones in the presence of nearby strong tones, resolvability of similar-strength nearby tones, resolvability of shifting tones, and biases in estimating the parameters of any of the aforementioned signals.

For practicality, the data we process are N uniformly spaced samples of the observed signal. For convenience, N is highly composite, and we will assume N is even. The harmonic estimates we obtain through the discrete Fourier transform (DFT) are N uniformly spaced samples of the associated periodic spectra. This approach is elegant and attractive when the processing scheme is cast as a spectral decomposition in an N-dimensional orthogonal vector space [2]. Unfortunately, in many practical situations, to obtain meaningful results this elegance must be compromised. One such compromise consists of applying windows to the sampled data set, or equivalently, smoothing the spectral samples.

The two operations to which we subject the data are sampling and windowing. These operations can be performed in either order. Sampling is well understood, windowing is less so, and sampled windows for DFT's significantly less so! We will address the interacting considerations of window selection in harmonic analysis and examine the special considerations related to sampled windows for DFT's.

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II. Harmonic Analysis of Finite-Extent Data and the DFT

Harmonic analysis of finite-extent data entails the projection of the observed signal on a basis set spanning the observation interval [1], [3]. Anticipating the next paragraph, we define T seconds as a convenient time interval and NT seconds as the observation interval. The sines and cosines with periods equal to an integer submultiple of NT seconds form an orthogonal basis set for continuous signals extending over NT seconds. These are defined as

$$\left.\begin{aligned} &\cos\left[\frac{2\pi}{NT}kt\right] \\ &\sin\left[\frac{2\pi}{NT}kt\right]\end{aligned}\right\}\quad \begin{aligned} &k = 0, 1, \cdots, N-1, N, N+1, \cdots \\ &0 \leqslant t < NT. \end{aligned} \tag{1}$$

We observe that by defining a basis set over an ordered index k, we are defining the spectrum over a line (called the frequency axis) from which we draw the concepts of bandwidth and of frequencies close to and far from a given frequency (which is related to resolution).

For sampled signals, the basis set spanning the interval of NT seconds is identical with the sequences obtained by uniform samples of the corresponding continuous spanning set up to the index $N/2$,

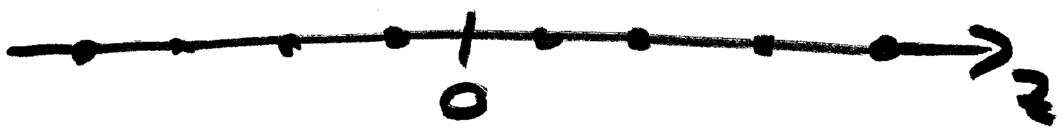
$$\left.\begin{aligned} &\cos\left[\frac{2\pi}{NT}knT\right] = \cos\left[\frac{2\pi}{N}kn\right] \\ &\sin\left[\frac{2\pi}{NT}knT\right] = \sin\left[\frac{2\pi}{N}kn\right]\end{aligned}\right\}\quad \begin{aligned} &k = 0, 1, \cdots, N/2 \\ &n = 0, 1, \cdots, N-1. \end{aligned} \tag{2}$$

We note here that the trigonometric functions are unique in that uniformly spaced samples (over an integer number of periods) form orthogonal sequences. Arbitrary orthogonal functions, similarly sampled, do not form orthogonal sequences. We also note that an interval of length NT seconds is not the same as the interval covered by N samples separated by intervals of T seconds. This is easily understood when we

8

FREQ.-SAMPLING FOR UNIFORMLY-SPACED ARRAYS

RECALL: $$B(\psi) = I(\psi) = e^{-j\left(\frac{N-1}{2}\right)\psi} \Bigg(\underbrace{\sum_{n=0}^{N-1} w_n e^{-jn\psi}}_{\text{DTFT OF } w_n \text{ (WEIGHT SEQUENCE)}} \Bigg)^{*}$$

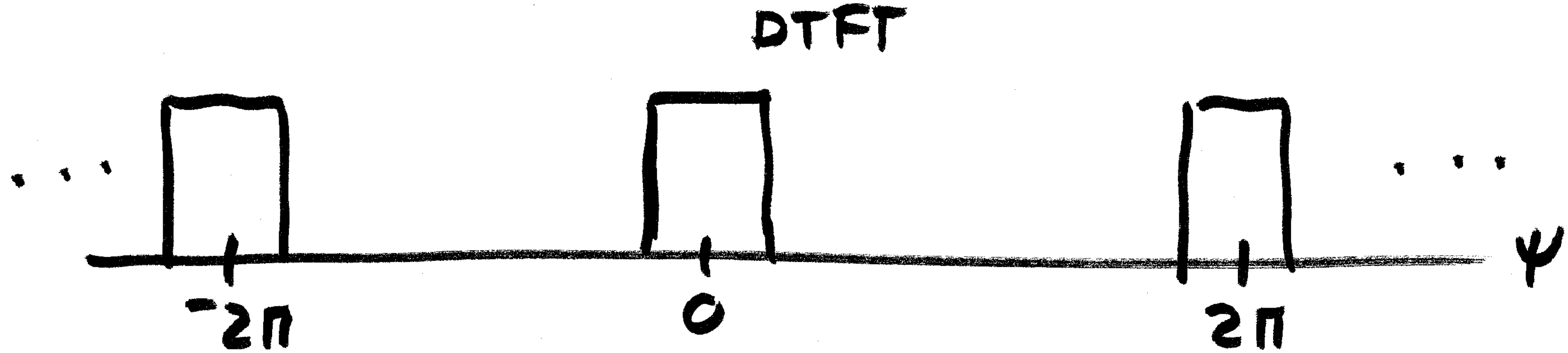


ARRAY CENTERED AT $z = 0$

REARRANGE:

$$B^{*}(\psi)\, e^{-j\left(\frac{N-1}{2}\right)\psi} = \sum_{n=0}^{N-1} w_n e^{-jn\psi}$$

DTFT'S ARE PERIODIC W/ PERIOD 2π



CAN WE GET w_n BY SAMPLING THIS DESIRED DTFT? YES $\longrightarrow$ DFT

DFT: SAMPLES EQUALLY SPACED 0 TO 2π

VAN TREES: N EQUALLY SPACED SAMPLES FROM $-\pi$ TO $+\pi$

$$\psi_k = \left(k - \left(\frac{N-1}{2}\right)\right)\frac{2\pi}{N} \qquad k = 0, \ldots, N-1$$

$$B(k) = B^*(\psi_k)\, e^{-j\left(\frac{N-1}{2}\right)\psi_k} = \sum_{n=0}^{N-1} w_n\, e^{-jn\psi_k}$$

$$B(k) = \sum_{n=0}^{N-1} w_n\, e^{-jn\left(k-\left(\frac{N-1}{2}\right)\right)\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} \underbrace{w_n\, e^{+jn\left(\frac{N-1}{N}\right)\pi}}_{b_n}\, e^{-jnk\frac{2\pi}{N}}$$

$$B(k) = \sum_{n=0}^{N-1} b_n\, e^{-j\frac{2\pi}{N}nk}$$ ← LOOKS LIKE **DFT**!

GIVEN: $B(k)$ SAMPLES OF DESIRED PATTERN

WANT: $b_n \longrightarrow w_n = b_n e^{-jn\left(\frac{N-1}{N}\right)\pi}$

IDFT:

$$b_n = \frac{1}{N} \sum_{k=0}^{N-1} B(k)\, e^{+j\frac{2\pi nk}{N}}$$

WILL GET $B(\psi) = B_{DESIRED}(\psi)\Big|_{\psi=\psi_k}$

SAMPLE POINT

COMPUTE $B(\psi) = \underline{w}^H \underline{v}(\psi)$

FOR ALL ψ TO GET PERFORMANCE

Figure 3.20: N=11

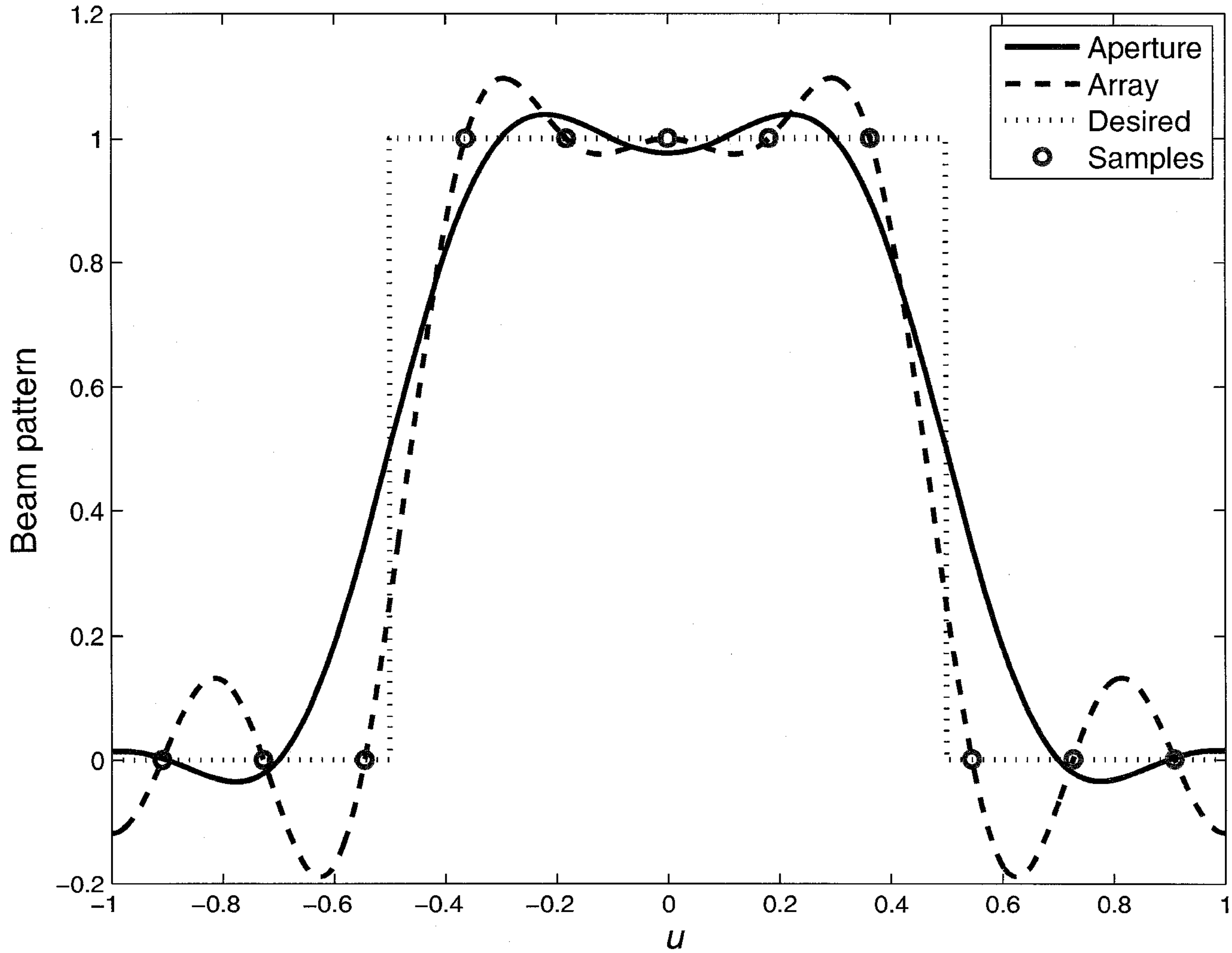
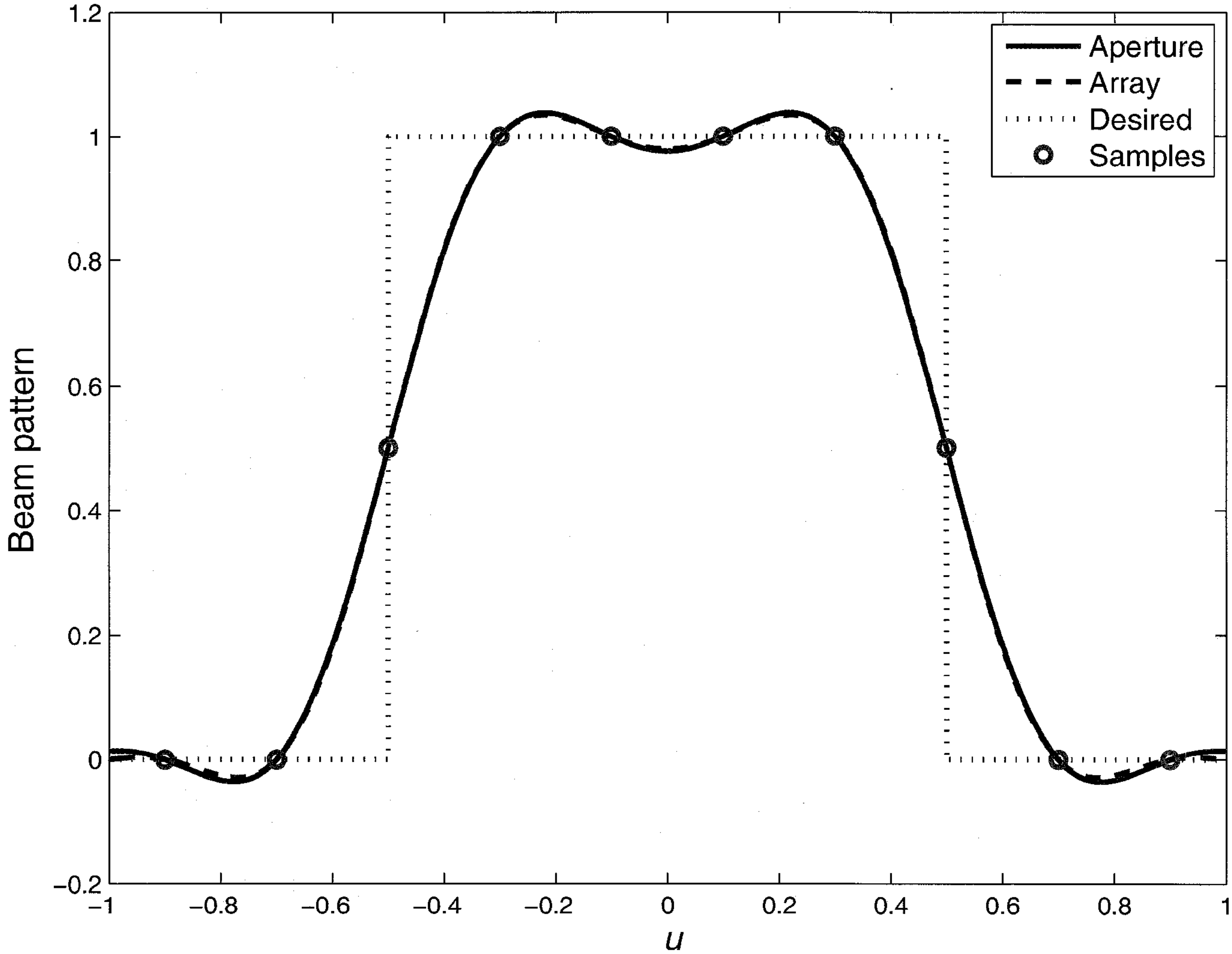


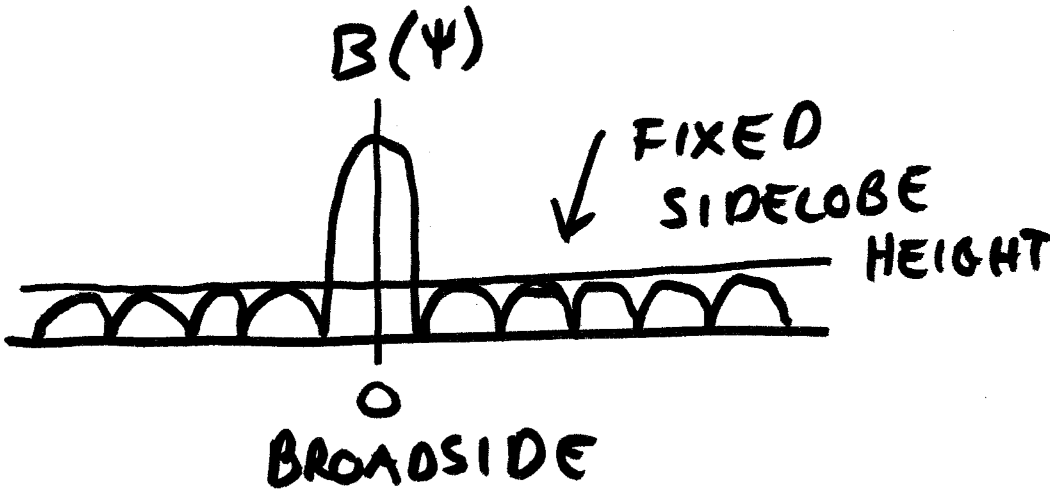
Figure 3.19: N=10



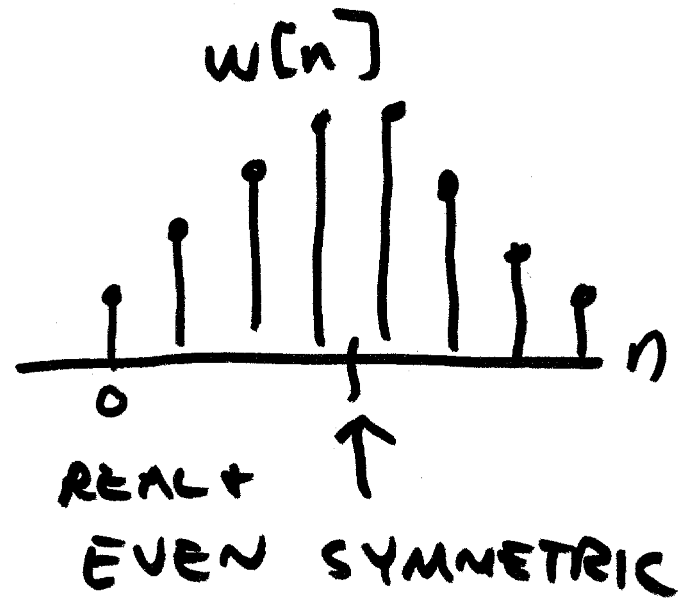
MINIMUM BEAMWIDTH FOR SPECIFIED SIDELOBE LEVEL

DOLPH-CHEBYCHEV APPROACH

WANT:



$\overset{F^{-1}}{\longleftrightarrow}$



$B(\Psi)$ IS REAL + EVEN

$$B(\Psi) = \underline{w}^H \underline{v}(\Psi) = \sum_{n=0}^{N-1} w_n^* e^{j\Psi\left(n - \left(\frac{N-1}{2}\right)\right)}$$

IF N = ODD

$$m = n - \left(\frac{N-1}{2}\right)$$

$$B(\psi) = \sum_{m=-\left(\frac{N-1}{2}\right)}^{+\left(\frac{N-1}{2}\right)} a_m e^{j\psi m} \qquad a_m = w_n \Big|_{n = m + \frac{N-1}{2}}$$

ASSUMING a_m REAL & EVEN

$$B(\psi) = a_0 + 2\sum_{m=1}^{\frac{N-1}{2}} a_m \cos(m\psi)$$

IF N = EVEN

$$B(\psi) = 2\sum_{m=1}^{N/2} a_m \cos\left(\left(m - \tfrac{1}{2}\right)\psi\right)$$

REWRITING:

ODD $$B(\Psi) = \sum_{k=0}^{\frac{N-1}{2}} \alpha_k \cos\left(2k \frac{\Psi}{2}\right)$$

EVEN $$B(\Psi) = \sum_{k=1}^{N/2} \alpha_k \cos\left((2k-1)\frac{\Psi}{2}\right)$$

$$\alpha_k = \begin{cases} 2a_k & k \neq 0 \\ a_k & k = 0 \end{cases}$$

DOLPH-CHEBYCHEV:

$B(\Psi)$ AS POLYNOMIAL IN $\cos(\Psi/2)$ OF ORDER $N-1$

LOOK AT TERM LIKE $\cos\left(M\frac{\Psi}{2}\right)$

$$\cos\left(M\frac{\Psi}{2}\right) = \mathrm{Re}\left\{\left(e^{j\frac{\Psi}{2}}\right)^{M}\right\}$$

$$= \mathrm{Re}\left\{\left[\cos\left(\frac{\Psi}{2}\right) + j\sin\left(\frac{\Psi}{2}\right)\right]^{M}\right\}$$

<u>BINOMIAL THM:</u>

$$(a+b)^{M} = \sum_{\ell=0}^{M} \binom{M}{\ell} a^{M-\ell} b^{M}$$

$$\binom{M}{\ell} = \frac{M!}{\ell!\,(M-\ell)!}$$

EX) $M=2$

$$\cos\left(2\left(\frac{\Psi}{2}\right)\right) = Re\left\{\left[\cos\left(\frac{\Psi}{2}\right) + j\sin\left(\frac{\Psi}{2}\right)\right]^2\right\}$$

$$= Re\left\{\cos^2\left(\frac{\Psi}{2}\right) + 2j\cos\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Psi}{2}\right) - \sin^2\left(\frac{\Psi}{2}\right)\right\}$$

$$= \cos^2\left(\frac{\Psi}{2}\right) - \sin^2\left(\frac{\Psi}{2}\right)$$

$$= \cos^2\left(\frac{\Psi}{2}\right) - \left(1 - \cos^2\left(\frac{\Psi}{2}\right)\right)$$

$$= 2\cos^2\left(\frac{\Psi}{2}\right) - 1 = \cos\left(2\left(\frac{\Psi}{2}\right)\right)$$

$B(\Psi)$ = $N-1$ ORDER POLYNOMIAL IN POWERS OF $\cos\left(\frac{\Psi}{2}\right)$

$$\cos\left(n\frac{\psi}{2}\right) = T_n(x)\Big|_{x=\cos\left(\frac{\psi}{2}\right)}$$

$T_n \triangleq$ nth CHEBYCHEV POLYNOMIAL

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$\vdots$

RECURSION:

$n \geq 2$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

$$T_M(x) = \begin{cases} \cos(M\cos^{-1}(x)) & |x| \le 1 \\ \cosh(M\cosh^{-1}(x)) & x > 1 \\ (-1)^M \cosh(M\cosh^{-1}(x)) & x < -1 \end{cases}$$

PROPERTIES:

* $T_M(x)$ HAS M REAL ROOTS IN INTERVAL $|x| < 1$

$$T_M(x) = \cos\left(M\frac{\Psi}{2}\right)\Big|_{\cos\left(\frac{\Psi}{2}\right) = x}$$

ROOTS: $\cos\left(M\frac{\Psi}{2}\right) = 0$

$$M\frac{\Psi}{2} = \frac{\pi}{2}(2p-1) \qquad p = 1, \ldots, M$$

ROOTS IN ψ SPACE:

$$\psi = \pi\left(\frac{2p-1}{M}\right)$$

IN x-space ROOTS ARE:

$$x_p = \cos\left((2p-1)\frac{\pi}{2M}\right)$$

PROPERTIES:

* $T_M(x)$ HAS ALTERNATING MINIMA + MAXIMA

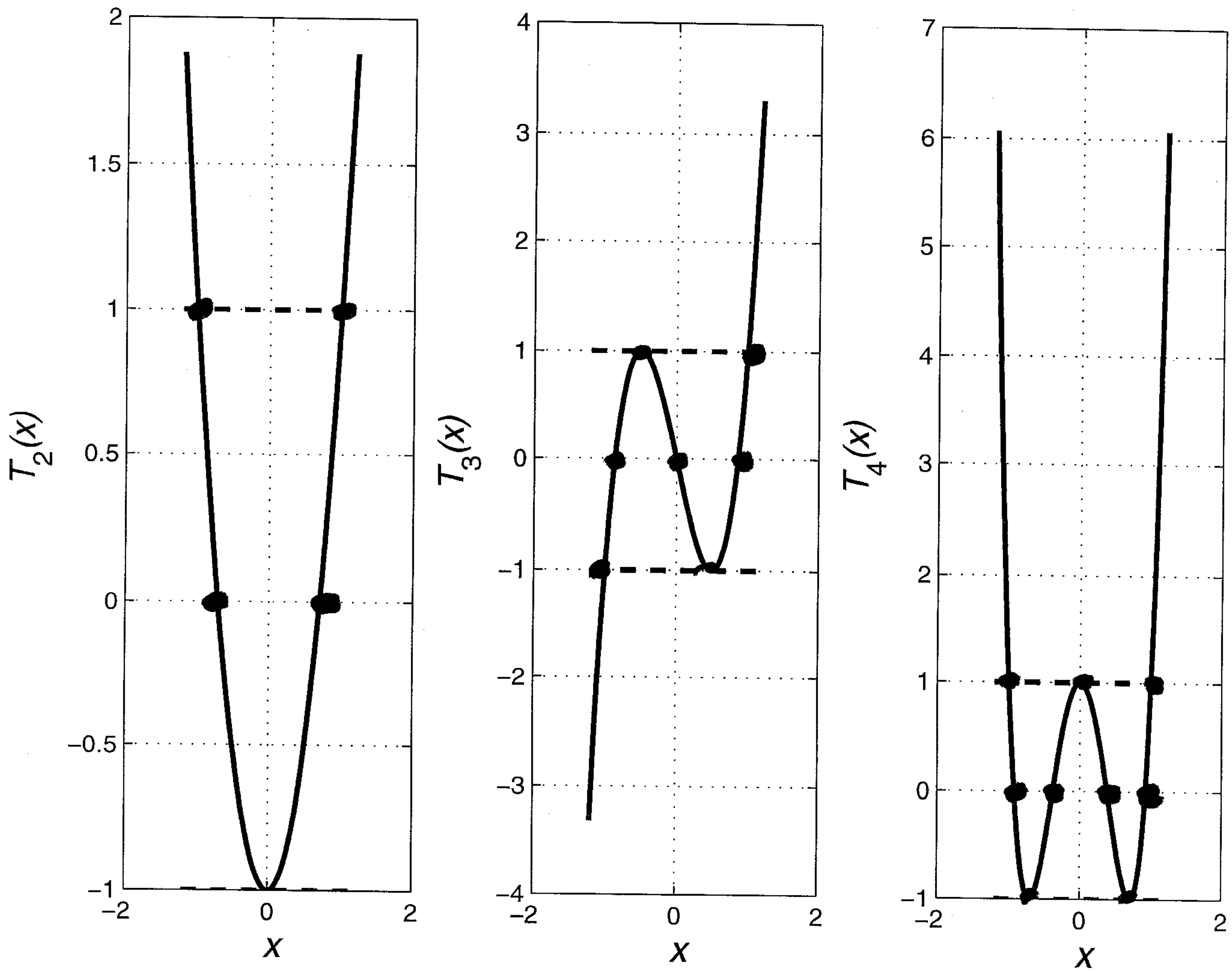
$$|T_{MIN}| = |T_{MAX}| = 1$$

EQUIRIPPLE IN x VARIABLE

* AT $x = \pm 1$, $|T_M(x)|_{x=\pm 1} = 1$

$x > 1$ $|T_M(x)| > 1$

FIGURE 3.23



DOLPH-CHEBY IDEA:

- $B(\psi)$ IS AN $N-1$ ORDER POLYNOMIAL IN $\cos\left(\frac{\psi}{2}\right)$
- SET $B(\psi)$ = CHEBYCHEV POLYNOMIAL SO WE GET EQUIRIPPLE PROPERTY

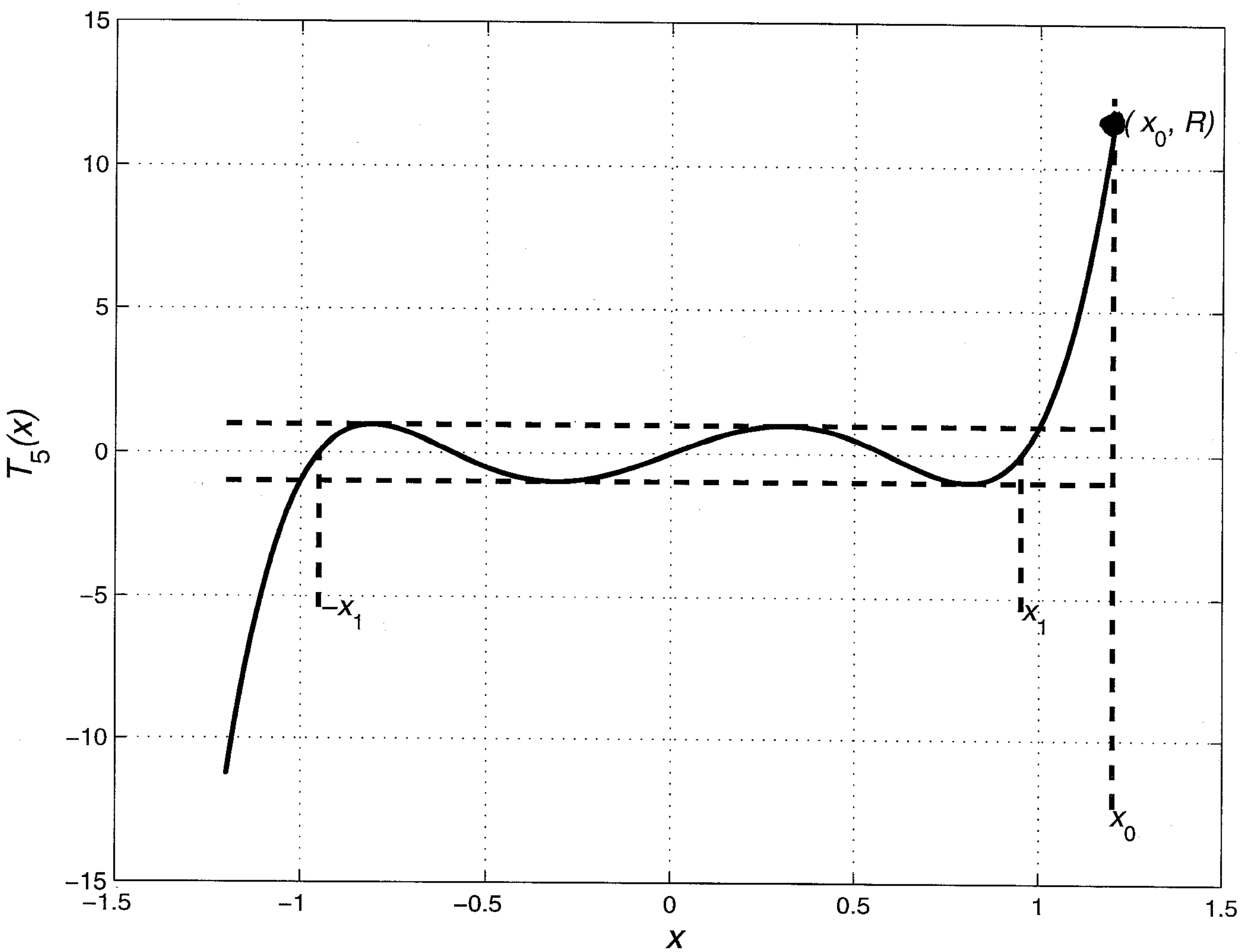
DEFINE A MAPPING SO THAT $B(\psi)$ MAXIMUM (M.L. MAX) CORRESPONDS TO A PT $x_0 > 1$

$$R = \frac{\text{M.L. MAX}}{\text{SIDELOBE LEVEL}}$$

SOLVE FOR x_0:

$$T_{N-1}(x_0) = R = \cosh\left((N-1)\cosh^{-1}(x_0)\right)$$

FIGURE 3.24



$$x_0 = \cosh\left(\frac{1}{N-1}\cosh^{-1}(R)\right)$$

RECALL MAPPING:

$$x = \cos\left(\frac{\psi}{2}\right)$$

RESTRICTS $-1 \le x \le +1$

ADJUST MAPPING:

$$q = \frac{x}{x_0} = \cos\left(\frac{\psi}{2}\right)$$

($q = w$ IN TEXTBOOK)

$$x = x_0 \cos\left(\frac{\psi}{2}\right)$$

BEAMPATTERN: $B(\psi) = \frac{1}{R}\, T_{N-1}\left(x_0 \cos\left(\frac{\psi}{2}\right)\right)$

HOW TO FIND WEIGHTS GIVEN $B(\psi)$ = CHEBY POLY.

$$\underline{w}^H \underline{v}(\psi)\Big|_{\psi=0} = 1$$ MAINLOBE CONSTRAINT

$$\underline{w}^H \underline{v}(\psi_p) = 0$$ N-1 ZEROS (NULLS IN BEAMPATTERN)

MATRIX FORM:

$$\underline{w}^H V = [1\ 0\ 0 \cdots 0]$$

OR $$V^H \underline{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} | & | & & | \\ \underline{v}(0) & \underline{v}(\psi_1) & \cdots & v(\psi_p) \\ | & | & & | \end{bmatrix}$$

SOLVING:

$$\underline{w} = (V^H)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

TEXT CALLS THIS $\underline{e}_1$

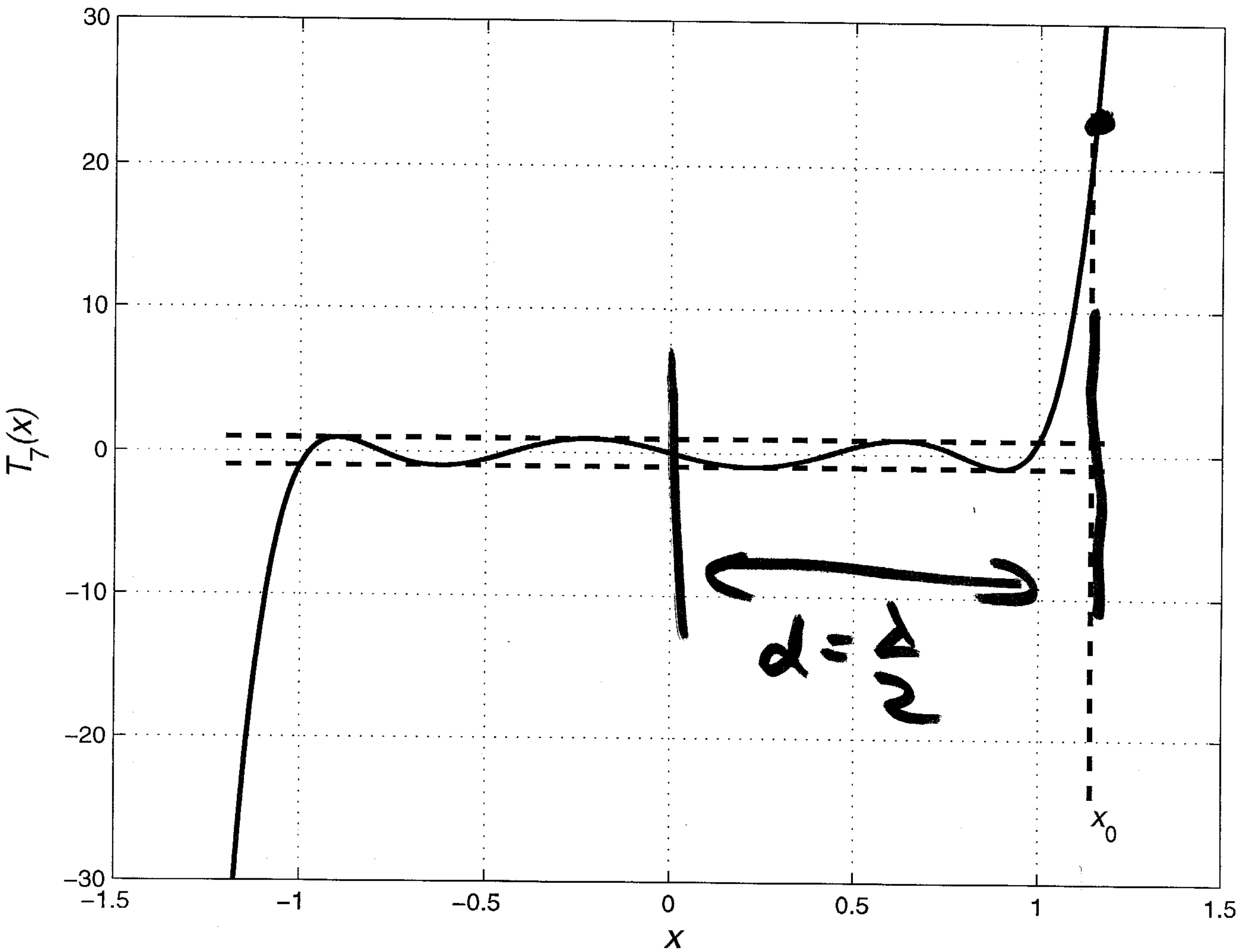
KNOW: WHERE ARE ψ_p's? NULLS

ORIGINAL SPACE: $\psi = \frac{\pi}{N-1}(2p-1) \quad p = 1, \ldots, M$

$$x_p = \cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)$$

NEW SPACE: $q_p = \frac{x}{x_0} = \frac{1}{x_0}\cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)$

FIGURE 3.25



NEW ZEROS: $\Psi_p = 2\cos^{-1}(q_p)$

$$\Psi_p = 2\cos^{-1}\left(\frac{1}{x_0}\cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)\right) \quad p=1,\ldots N-1$$

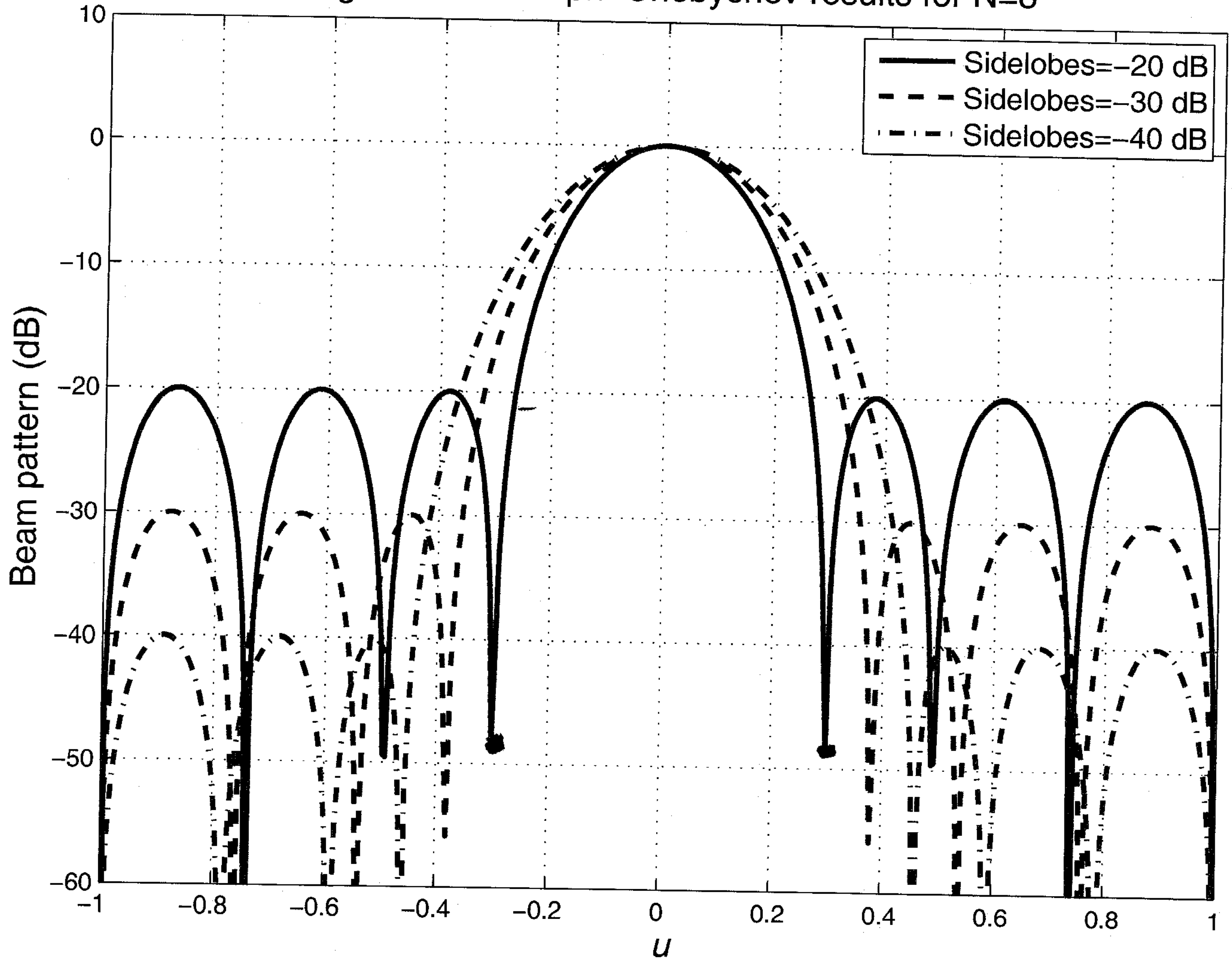
MAPPING:

$$B(\Psi) = \frac{1}{R} T_{N-1}\Big(\underbrace{x_0\cos\left(\tfrac{\Psi}{2}\right)}\Big) = T_{N-1}(x)$$

$$x = x_0 \cos\left(\frac{\Psi}{2}\right)$$

θ	0	$\pi/2$	π
$\Psi = \frac{2\pi}{\lambda} d\cos\theta$	$\frac{2\pi d}{\lambda}$	0	$-\frac{2\pi d}{\lambda}$
$x = x_0\cos\left(\frac{\Psi}{2}\right)$	$x_0\cos\left(\frac{\pi d}{\lambda}\right)$	x_0	$x_0\cos\left(\frac{\pi d}{\lambda}\right)$

Figure 3.26: Dolph–Chebychev results for N=8



MAPPING:

$d = \frac{\lambda}{4}$:

$$x_0 \cos\left(\frac{\pi \lambda/4}{\lambda}\right) \le x \le x_0$$

$$x_0 \cos\left(\frac{\pi}{4}\right) \le x \le x_0$$

$$.707\, x_0 \le x \le x_0$$

$d = \frac{\lambda}{2}$

$$x_0 \cos\left(\frac{\pi \lambda/2}{\lambda}\right) \le x \le x_0$$

$$0 \le x \le x_0$$

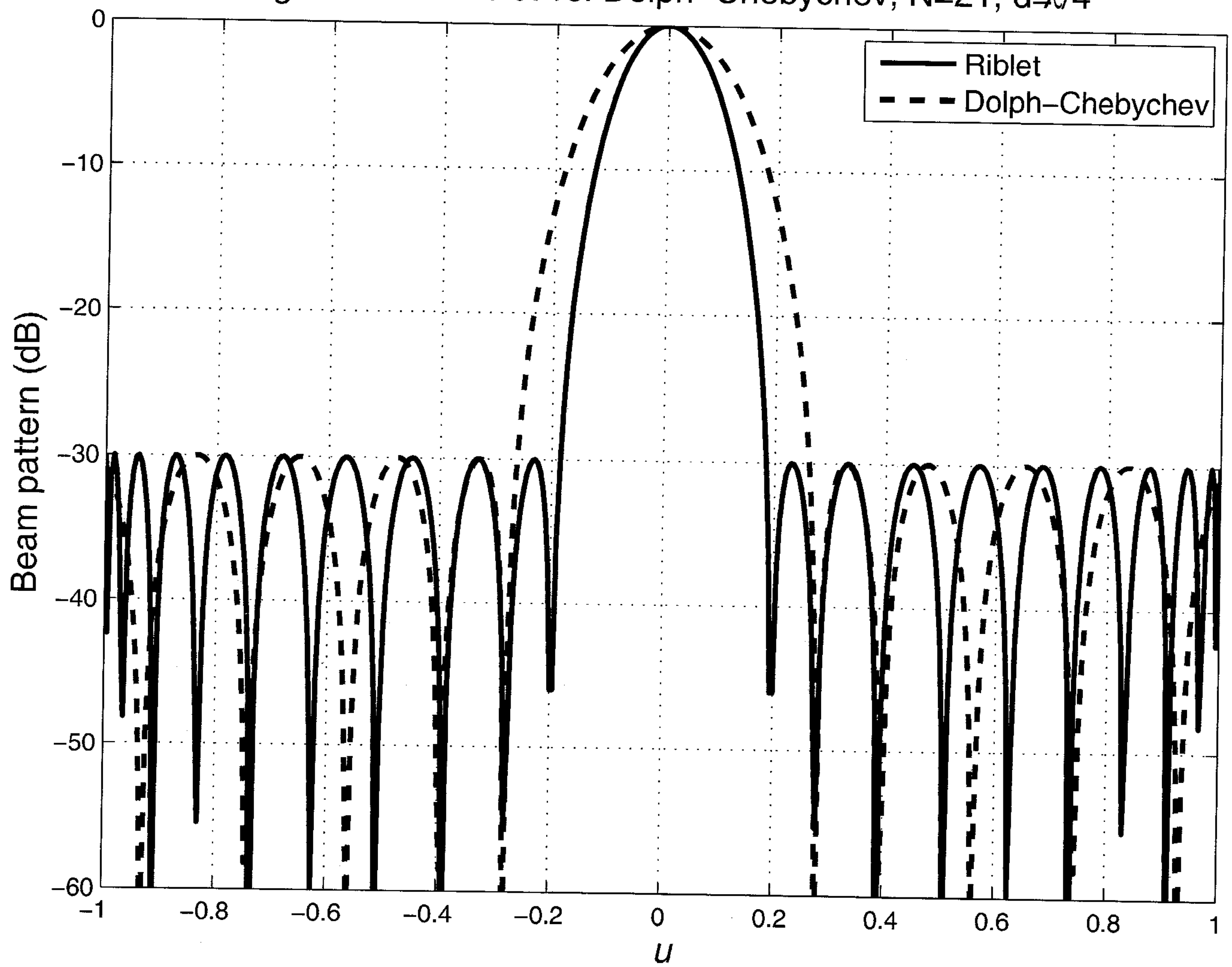
$d = \frac{3\lambda}{4}$

$$-.707\, x_0 \le x \le x_0$$

$$\boxed{d \le \frac{\lambda}{\pi} \cos^{-1}\left(\frac{-1}{x_0}\right)}$$

$\leftarrow$ FOR MAPPING TO BE VALID (LEFT EDGE = -1)

Figure 3.28: Riblet vs. Dolph–Chebychev, N=21, d=λ/4



RIBLET - CHEBYCHEV

- USES EXPANSION IN TERMS OF $\cos(\Psi)$ INSTEAD $\cos\left(\frac{\Psi}{2}\right)$
- POLYNOMIAL OF ORDER $\frac{N-1}{2}$ (N ODD)
- MAPPING: $x = C_1 \cos\Psi + C_2$

SOLVE FOR C_1 + C_2 SO THAT FULL RANGE OF CHEBYCHEV POLYNOMIAL IS USED

$$-1 \leq x \leq x_0$$

$d = \frac{\lambda}{2}$ NO DIFFERENCE | $d < \frac{\lambda}{2}$ DIFFERENCE

VILLENEUVE

- START W/ ZEROS ASSOCIATED W/ UNIFORM WEIGHTING (WARM-UP)
- EXCHANGE $2\bar{n}$ OF THOSE NEAR BROADSIDE FOR $\approx$ CHEBYVEV ROOTS

=> RESULT: BETTER S.L. NEAR BROADSIDE

DECREASING S.L. FARTHER OUT

FIGURE 3.30

Roots in z–plane, N=21, $\bar{n}$=6

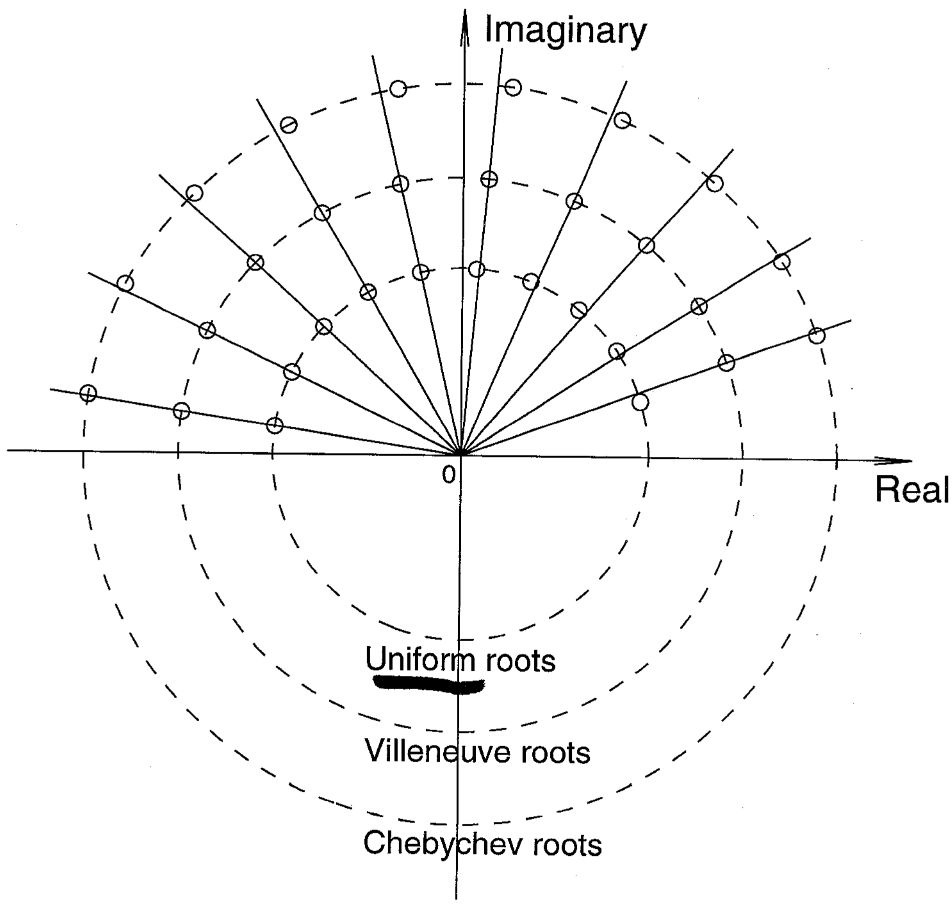


Figure 3.31: Villenueve: N=41, nbar=6

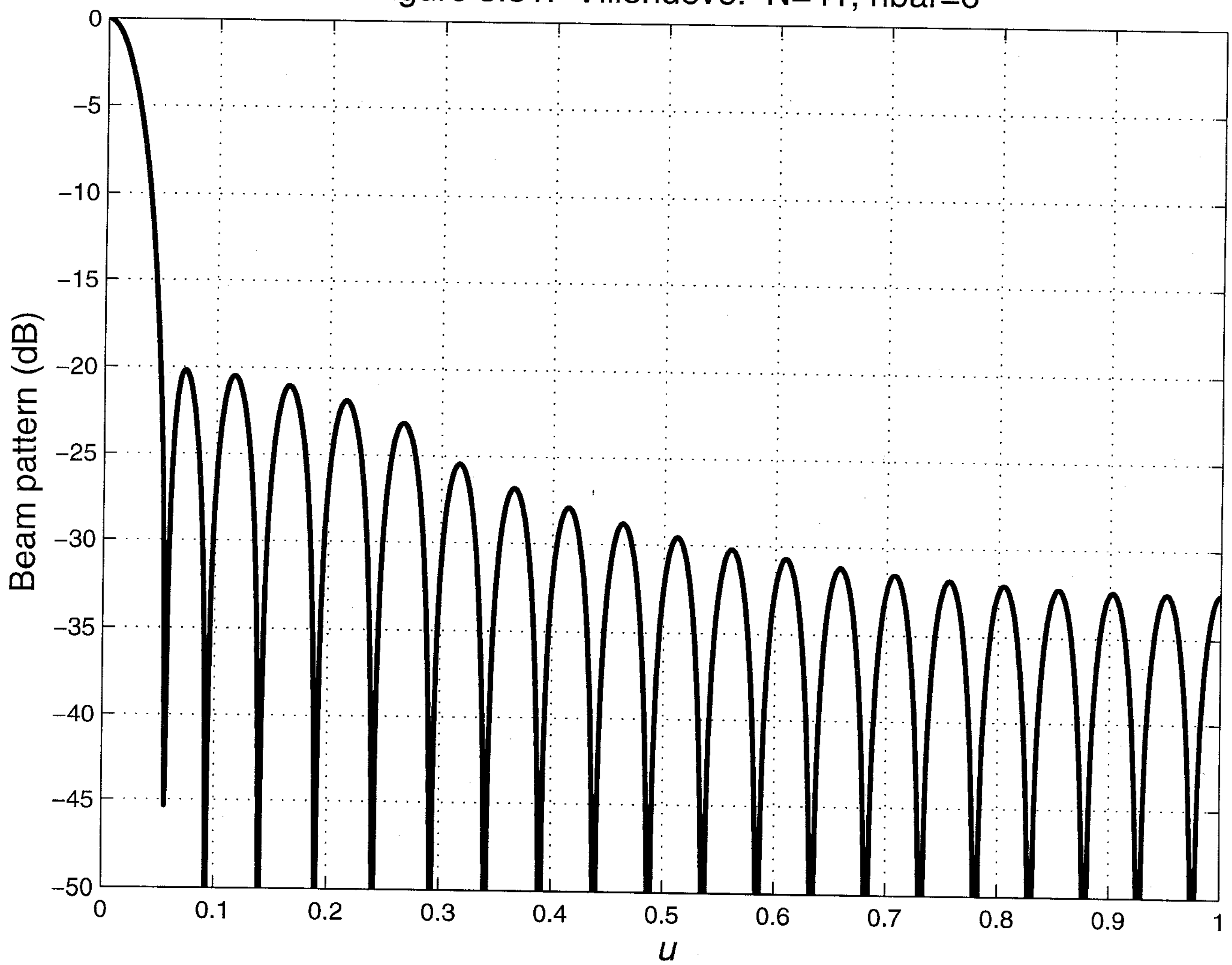


Figure 3.31: Villenueve: N=21, nbar=6