ECE 754

WARM-UP PROBLEM

GIVEN: $w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{OTHER} \end{cases}$

FIND DTFT OF $w[n]$ + SKETCH $|W(e^{j\omega})|$

FIND z-TRANS. OF $w[n]$

SKETCH POLES + ZEROS OF $W(z)$

WARM-UP

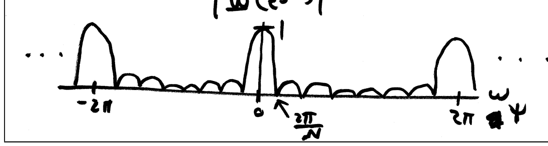
$$W(z) = \sum_n w[n] z^{-n} = \frac{1-z^{-N}}{1-z^{-1}}$$

$$W(e^{j\omega}) = \sum_n w[n] e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

? HOW DO z + FOURIER RELATE?

$$W(e^{j\omega}) = W(z)\Big|_{z=e^{j\omega}}$$ z ON UNIT CIRCLE

$|W(e^{j\omega})|$



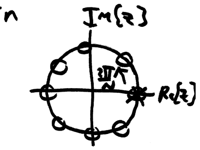
WARM-UP

$$W(z) = \frac{1-z^{-N}}{1-z^{-1}} \cdot \frac{z^N}{z^N}$$ POLES + ZEROS?

$$= \frac{z^N - 1}{z^{N-1}(z-1)}$$

ZEROS: $z^N = 1 = e^{j2\pi n}$

$z = e^{j\frac{2\pi n}{N}}$



POLES: $z^{N-1} = 0$

$z = 0$ N-1 POLES AT $z=0$

1 POLE AT $z=1$

CH. 3: DESIGN PROBLEM

CHOOSE $\underline{w}$ TO MEET CERTAIN CRITERIA

EX) MINIMIZE BEAMWIDTH (FILTER PASSBAND WIDTH)

CONTROL SIDELOBE LEVELS

UNCERTAINTY PRINCIPLE:

$$\sqrt{\overline{\Delta l^2}}\sqrt{\overline{\Delta k^2}} \ge \frac{1}{2}$$

NORMALIZED MEAN SQ. WIDTH OF APERTURE

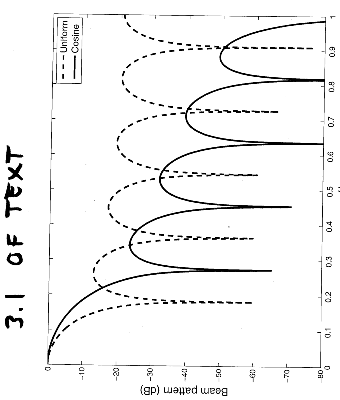
NORM. MEAN SQ WIDTH OF SPATIAL RESPONSE

FOCUS ON LINEAR, EQUALLY SPACED ARRAYS

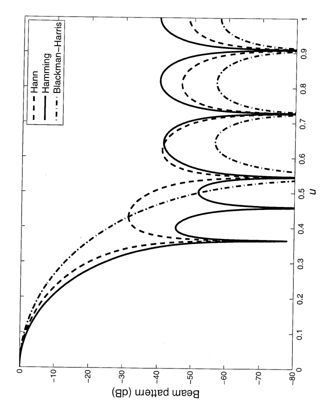
DESIGN METHODS:

- $\underline{w}$ = STANDARD WINDOW AS USED IN SPECTRAL ESTIMATION
- FREQUENCY SAMPLING
- MINIMUM BW FOR FIXED SIDELOBE LEVEL → DOLPH-CHEBYCHEV
- LEAST SQUARES
  MINIMIZE MSQ ERROR BETWEEN DESIRED + ACTUAL BEAM PATTERN
- MIN-MAX (PARKS-McCLELLAN)

3.1 OF TEXT



TRADEOFF: M.L. WIDTH
SIDELOBE HEIGHT



PROCEEDINGS OF THE IEEE, VOL. 66, NO. 1, JANUARY 1978

On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform

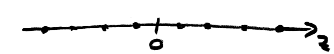
FREDRIC J. HARRIS, MEMBER, IEEE

I. INTRODUCTION

II. HARMONIC ANALYSIS OF FINITE-EXTENT DATA AND THE DFT

FREQ.-SAMPLING FOR UNIFORMLY-SPACED ARRAYS

RECALL: $$B(\psi) = \Upsilon(\psi) = e^{-j\left(\frac{N-1}{2}\right)\psi}\Big(\underbrace{\sum_{n=0}^{N-1} w_n e^{-jn\psi}}_{\text{DTFT OF } w_n \text{ (WEIGHT SEQUENCE)}}\Big)^*$$



ARRAY CENTERED AT $z=0$

REARRANGE:

$$B^*(\psi)\, e^{-j\left(\frac{N-1}{2}\right)\psi} = \sum_{n=0}^{N-1} w_n e^{-jn\psi}$$

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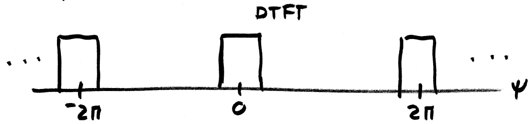
DTFT'S ARE PERIODIC W/ PERIOD 2π



CAN WE GET w_n BY SAMPLING THIS DESIRED DTFT? YES $\longrightarrow$ DFT

DFT: SAMPLES EQUALLY SPACED 0 TO 2π

VAN TREES: N EQUALLY SPACED SAMPLES FROM $-\pi$ TO $+\pi$

$$\psi_k = \left(k - \left(\frac{N-1}{2}\right)\right)\frac{2\pi}{N} \qquad k = 0, \dots, N-1$$

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$$B(k) = B^*(\psi_k)\, e^{-j\left(\frac{N-1}{2}\right)\psi_k} = \sum_{n=0}^{N-1} w_n e^{-jn\psi_k}$$

$$B(k) = \sum_{n=0}^{N-1} w_n e^{-jn\left(k - \left(\frac{N-1}{2}\right)\right)\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} \underbrace{w_n e^{+jn\left(\frac{N-1}{N}\right)\pi}}_{b_n} e^{-jnk\frac{2\pi}{N}}$$

$$B(k) = \sum_{n=0}^{N-1} b_n e^{-j\frac{2\pi}{N}nk}$$ $\leftarrow$ LOOKS LIKE DFT!

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GIVEN: $B(k)$ SAMPLES OF DESIRED PATTERN

WANT: $b_n \longrightarrow w_n = b_n e^{-jn\left(\frac{N-1}{N}\right)\pi}$

IDFT: $$b_n = \frac{1}{N}\sum_{k=0}^{N-1} B(k)\, e^{+j\frac{2\pi nk}{N}}$$

WILL GET $B(\psi) = B_{DESIRED}(\psi)\Big|_{\psi=\psi_k}$ SAMPLE POINT

COMPUTE $B(\psi) = \underline{w}^H \underline{v}(\psi)$ FOR ALL ψ TO GET PERFORMANCE

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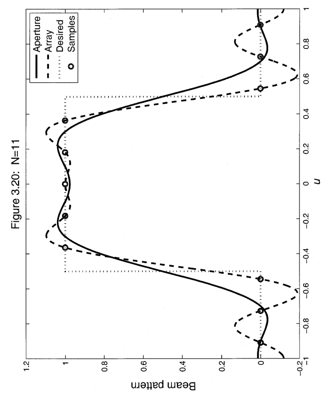

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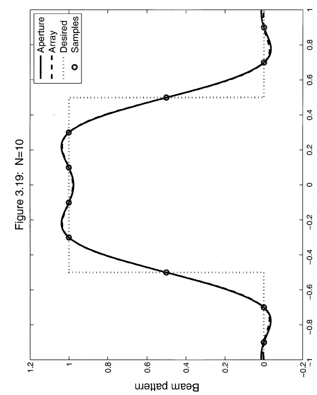

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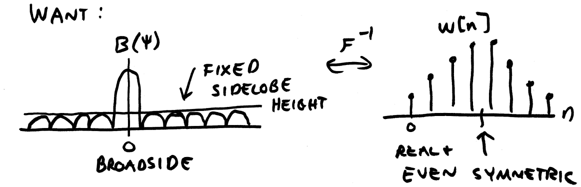
MINIMUM BEAMWIDTH FOR SPECIFIED SIDELOBE LEVEL

DOLPH-CHEBYCHEV APPROACH

WANT:



$B(\psi)$ IS REAL + EVEN

$$B(\psi) = \underline{w}^H \underline{v}(\psi) = \sum_{n=0}^{N-1} w_n^* e^{j\psi\left(n - \left(\frac{N-1}{2}\right)\right)}$$

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IF N = ODD $\quad m = n - \left(\frac{N-1}{2}\right)$

$$B(\psi) = \sum_{m=-\left(\frac{N-1}{2}\right)}^{+\left(\frac{N-1}{2}\right)} a_m e^{j\psi m} \qquad a_m = w_n\Big|_{n=m+\frac{N-1}{2}}$$

ASSUMING a_m REAL + EVEN

$$B(\psi) = a_0 + 2\sum_{m=1}^{\frac{N-1}{2}} a_m \cos(m\psi)$$

IF N = EVEN

$$B(\psi) = 2\sum_{m=1}^{N/2} a_m \cos\left(\left(m - \tfrac{1}{2}\right)\psi\right)$$

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REWRITING:

ODD $$B(\psi) = \sum_{k=0}^{\frac{N-1}{2}} \alpha_k \cos\left(2k\frac{\psi}{2}\right)$$

EVEN $$B(\psi) = \sum_{k=1}^{N/2} \alpha_k \cos\left((2k-1)\frac{\psi}{2}\right)$$

$$\alpha_k = \begin{cases} 2a_k & k \neq 0 \\ a_k & k = 0 \end{cases}$$

DOLPH-CHEBYCHEV:
$B(\psi)$ AS POLYNOMIAL IN $\cos(\psi/2)$ OF ORDER $N-1$

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LOOK AT TERM LIKE $\cos\left(m\frac{\psi}{2}\right)$

$$\cos\left(m\frac{\psi}{2}\right) = Re\left\{\left(e^{j\frac{\psi}{2}}\right)^m\right\}$$

$$= Re\left\{\left[\cos\left(\frac{\psi}{2}\right) + j\sin\left(\frac{\psi}{2}\right)\right]^m\right\}$$

BINOMIAL THM:

$$(a+b)^m = \sum_{l=0}^{m} \binom{m}{l} a^{m-l} b^{m}$$

$$\binom{m}{l} = \frac{m!}{l!\,(m-l)!}$$

EX) $M = 2$

$$\cos\left(2\left(\tfrac{\psi}{2}\right)\right) = \mathrm{Re}\left\{\left[\cos\left(\tfrac{\psi}{2}\right) + j\sin\left(\tfrac{\psi}{2}\right)\right]^2\right\}$$

$$= \mathrm{Re}\left\{\cos^2\left(\tfrac{\psi}{2}\right) + 2j\cos\left(\tfrac{\psi}{2}\right)\sin\left(\tfrac{\psi}{2}\right) - \sin^2\left(\tfrac{\psi}{2}\right)\right\}$$

$$= \cos^2\left(\tfrac{\psi}{2}\right) - \sin^2\left(\tfrac{\psi}{2}\right)$$

$$= \cos^2\left(\tfrac{\psi}{2}\right) - \left(1 - \cos^2\left(\tfrac{\psi}{2}\right)\right)$$

$$= 2\cos^2\left(\tfrac{\psi}{2}\right) - 1 = \cos\left(2\left(\tfrac{\psi}{2}\right)\right)$$

$B(\psi)$ = $N-1$ ORDER POLYNOMIAL IN POWERS OF $\cos\left(\frac{\psi}{2}\right)$

$$\cos\left(M\tfrac{\psi}{2}\right) = T_M(x)\Big|_{x = \cos\left(\frac{\psi}{2}\right)}$$

$T_M \triangleq$ Mth CHEBYCHEV POLYNOMIAL

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$\vdots$

RECURSION: $M \geq 2$

$$T_M(x) = 2xT_{M-1}(x) - T_{M-2}(x)$$

$$T_M(x) = \begin{cases} \cos(M\cos^{-1}(x)) & |x| \leq 1 \\ \cosh(M\cosh^{-1}(x)) & x > 1 \\ (-1)^M \cosh(M\cosh^{-1}(x)) & x < -1 \end{cases}$$

PROPERTIES:

* $T_M(x)$ HAS M REAL ROOTS IN INTERVAL $|x| < 1$

$$T_M(x) = \cos\left(M\tfrac{\psi}{2}\right)\Big|_{\cos\left(\frac{\psi}{2}\right) = x}$$

ROOTS: $\cos\left(M\frac{\psi}{2}\right) = 0$

$$M\frac{\psi}{2} = \frac{\pi}{2}(2p-1) \qquad p = 1, \ldots, M$$

ROOTS IN ψ SPACE:

$$\psi = \pi\left(\frac{2p-1}{M}\right)$$

IN x-SPACE ROOTS ARE:

$$x_p = \cos\left((2p-1)\frac{\pi}{2M}\right)$$

PROPERTIES:

* $T_M(x)$ HAS ALTERNATING MINIMA + MAXIMA

$$|T_{MIN}| = |T_{MAX}| = 1$$

EQUIRIPPLE IN x VARIABLE

* AT $x = \pm 1$, $|T_M(x)|_{x=\pm1} = 1$

$x > 1$ $|T_M(x)| > 1$

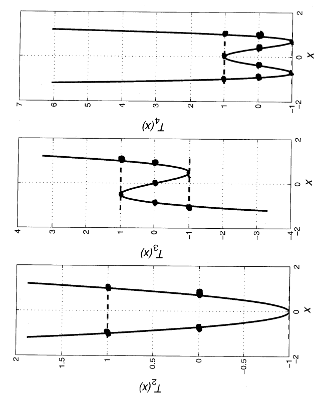


FIGURE 3.23

DOLPH-CHEBY IDEA:

* $B(\psi)$ IS AN $N-1$ ORDER POLYNOMIAL IN $\cos\left(\frac{\psi}{2}\right)$
* SET $B(\psi)$ = CHEBYCHEV POLYNOMIAL SO WE GET EQUIRIPPLE PROPERTY

DEFINE A MAPPING SO THAT $B(\psi)$ MAXIMUM (M.L. MAX) CORRESPONDS TO A PT $x_0 > 1$

$$R = \frac{\text{M.L. MAX}}{\text{SIDELOBE LEVEL}}$$

SOLVE FOR x_0:

$$T_{N-1}(x_0) = R = \cosh\left((N-1)\cosh^{-1}(x_0)\right)$$

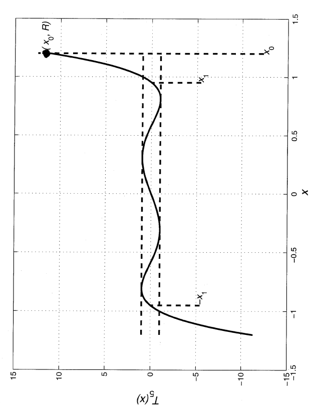


FIGURE 3.24

$$x_0 = \cosh\left(\frac{1}{N-1}\cosh^{-1}(R)\right)$$

RECALL MAPPING:

$$x = \cos\left(\frac{\psi}{2}\right)$$

RESTRICTS $-1 \leq x \leq +1$

ADJUST MAPPING:

$$q = \frac{x}{x_0} = \cos\left(\frac{\psi}{2}\right) \qquad \left(\begin{array}{c} q = w \\ \text{IN TEXTBOOK} \end{array}\right)$$

$$x = x_0\cos\left(\frac{\psi}{2}\right)$$

BEAMPATTERN: $B(\psi) = \frac{1}{R}T_{N-1}\left(x_0\cos\left(\frac{\psi}{2}\right)\right)$

HOW TO FIND WEIGHTS GIVEN $B(\psi)$ = CHEBY POLY.

$\underline{w}^H \underline{v}(\psi)\Big|_{\psi=0} = 1$ MAINLOBE CONSTRAINT

$\underline{w}^H \underline{v}(\psi_p) = 0$ $N-1$ ZEROS (NULLS IN BEAMPATTERN)

MATRIX FORM:

$$\underline{w}^H V = [1\ 0\ 0 \cdots 0]$$

OR

$$V^H \underline{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} | & | & & | \\ \underline{v}(0) & \underline{v}(\psi_1) & \cdots & \underline{v}(\psi_p) \\ | & | & & | \end{bmatrix}$$

SOLVING:

$$\underline{w} = (V^H)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

TEXT CALLS THIS $\underline{e}_1$

KNOW: WHERE ARE ψ_p's? NULLS

ORIGINAL SPACE: $\psi = \frac{\pi}{N-1}(2p-1) \quad p = 1, \dots, M$

$$x_p = \cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)$$

NEW SPACE: $q_p = \frac{x}{x_0} = \frac{1}{x_0}\cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)$

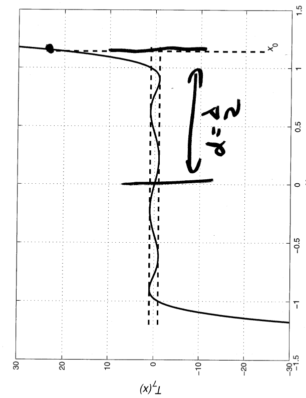


FIGURE 3.25

NEW ZEROS: $\psi_p = 2\cos^{-1}(q_p)$

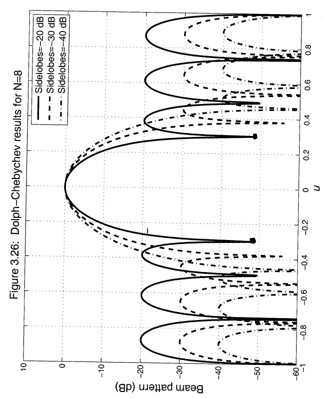
$$\psi_p = 2\cos^{-1}\left(\frac{1}{x_0}\cos\left(\frac{\pi}{2}\left(\frac{2p-1}{N-1}\right)\right)\right) \quad p = 1, \dots N-1$$

MAPPING:

$$B(\psi) = \frac{1}{R} T_{N-1}\Big(\underbrace{x_0\cos\left(\tfrac{\psi}{2}\right)}\Big) = T_{N-1}(x)$$

$$x = x_0 \cos\left(\frac{\psi}{2}\right)$$

θ	0	$\pi/2$	π
$\psi = \frac{2\pi}{\lambda} d\cos\theta$	$\frac{2\pi d}{\lambda}$	0	$-\frac{2\pi d}{\lambda}$
$X = x_0\cos\left(\frac{\psi}{2}\right)$	$x_0\cos\left(\frac{\pi d}{\lambda}\right)$	x_0	$x_0\cos\left(\frac{\pi d}{\lambda}\right)$



MAPPING:

$d = \frac{\lambda}{4}$: $\quad x_0\cos\left(\frac{\pi \lambda/4}{\lambda}\right) \le x \le x_0$

$$x_0 \cos\left(\frac{\pi}{4}\right) \le x \le x_0$$

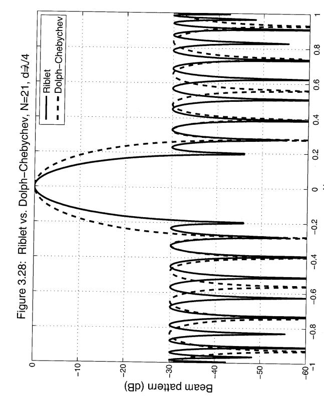
$$.707\, x_0 \le x \le x_0$$

$d = \frac{\lambda}{2}$ $\quad x_0\cos\left(\frac{\pi \lambda/2}{\lambda}\right) \le x \le x_0$

$$0 \le x \le x_0$$

$d = \frac{3\lambda}{4}$ $\quad -.707\, x_0 \le x \le x_0$

$\boxed{d \le \frac{\lambda}{\pi}\cos^{-1}\left(\frac{-1}{x_0}\right)}$ ← FOR MAPPING TO BE VALID | LEFT EDGE = -1



RIBLET - CHEBYCHEV

- USES EXPANSION IN TERMS OF $\cos(\psi)$ INSTEAD $\cos\left(\frac{\psi}{2}\right)$
- POLYNOMIAL OF ORDER $\frac{N-1}{2}$ (N ODD)
- MAPPING: $x = c_1\cos\psi + c_2$

SOLVE FOR c_1 + c_2 SO THAT FULL RANGE OF CHEBYCHEV POLYNOMIAL IS USED

$$-1 \le x \le x_0$$

$d = \frac{\lambda}{2}$ NO DIFFERENCE | $d < \frac{\lambda}{2}$ DIFFERENCE

VILLENEUVE

- START W/ ZEROS ASSOCIATED W/ UNIFORM WEIGHTING (WARM-UP)
- EXCHANGE $2\bar{n}$ OF THOSE NEAR BROADSIDE FOR ≈ CHEBYCHEV ROOTS

=> RESULT: BETTER S.L. NEAR BROADSIDE

DECREASING S.L. FARTHER OUT

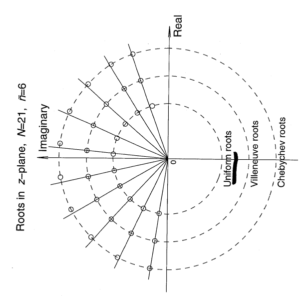


FIGURE 3.30

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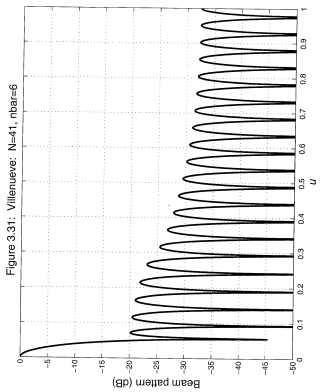

Figure 3.31: Villenueve: N=41, nbar=6

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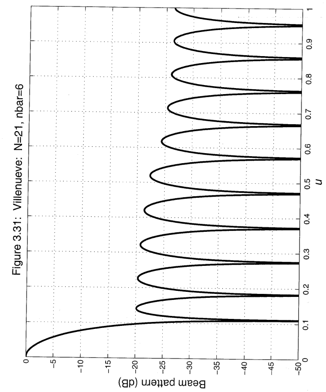

Figure 3.31: Villenueve: N=21, nbar=6