Optimum Beamforming

ECE 754 Supplemental Notes
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Signal and noise models

For this set of notes, we assume that the snapshot vector $\mathbf{x}$ consists of a planewave signal component plus a noise component, *i.e.*,

$$\mathbf{x} = \mathbf{x}_{\text{sig}} + \mathbf{n} = \tilde{b}_s \mathbf{v}_s + \mathbf{n}$$

where $\tilde{b}_s$ is the complex weight applied to the planewave replica $\mathbf{v}_s$ associated with the source. $\mathbf{n}$ is the noise vector. The statistics of the source and noise components are:

$$\mathbf{S}_{\text{sig}} = \mathcal{E}\{\mathbf{x}_{\text{sig}} \mathbf{x}_{\text{sig}}^{\text{H}}\} = \mathcal{E}\{\tilde{b}_s \tilde{b}_s^*\} \mathbf{v}_s \mathbf{v}_s^{\text{H}} = \sigma_s^2 \mathbf{v}_s \mathbf{v}_s^{\text{H}}$$

$$\mathbf{S}_{\mathbf{n}} = \mathcal{E}\{\mathbf{n}\mathbf{n}^{\text{H}}\}$$

Models continued...

When the noise consists of a single planewave interferer plus spatially white noise, the noise covariance matrix is:

$$\mathbf{S_n} = \sigma_1^2 \mathbf{v}_1 \mathbf{v}_1^{\mathrm{H}} + \sigma_w^2 \mathbf{I}$$

where the vector $\mathbf{v}_1$ is the replica for the planewave interferer and σ_1^2 is the power in the interferer.

We define the interferer-to-noise ratio (INR) as:

$$\sigma_I^2 = \frac{\sigma_1^2}{\sigma_w^2}$$

Number of array elements is denoted by N in these notes.

Beamformers

In equations below, $\mathbf{v}_m =$ replica vector for the steering direction

Conventional processor (uniformly weighted):

$$\mathbf{w}_{\mathrm{conv}} = \frac{\mathbf{v}_m}{N}$$

Minimum variance distortionless response processor:

$$\mathbf{w}_{\mathrm{mvdr}} = \underbrace{\frac{1}{\mathbf{v}_m^{\mathrm{H}} \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{v}_m}}_{\Lambda_{\mathrm{mvdr}}} \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{v}_m = \Lambda_{\mathrm{mvdr}} \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{v}_m$$

Minimum power distortionless response processor:

$$\mathbf{w}_{\mathrm{mpdr}} = \underbrace{\frac{1}{\mathbf{v}_m^{\mathrm{H}} \mathbf{S}_{\mathbf{x}}^{-1} \mathbf{v}_m}}_{\Lambda_{\mathrm{mpdr}}} \mathbf{S}_{\mathbf{x}}^{-1} \mathbf{v}_m = \Lambda_{\mathrm{mpdr}} \mathbf{S}_{\mathbf{x}}^{-1} \mathbf{v}_m$$

MVDR beampattern for a single interferer

Consider the case when the noise consists of a single planewave interferer plus spatially white noise. As derived in class (Lecture #8), the MVDR beampattern in this case can be written:

$$
\begin{aligned}
B_{\mathrm{mvdr}}(u_a) &= \mathbf{w}_{\mathrm{mvdr}}^{\mathrm{H}} \mathbf{v}_a \\
&= \frac{\Lambda_{\mathrm{mvdr}} N}{\sigma_w^2} \left[\rho_{ma} - \frac{N \sigma_I^2 \rho_{m1}}{1 + N \sigma_I^2} \rho_{1a} \right]
\end{aligned}
$$

where

$$
\rho_{ma} = \frac{\mathbf{v}_m^{\mathrm{H}} \mathbf{v}_a}{N} \qquad \rho_{m1} = \frac{\mathbf{v}_m^{\mathrm{H}} \mathbf{v}_1}{N} \qquad \rho_{1a} = \frac{\mathbf{v}_1^{\mathrm{H}} \mathbf{v}_a}{N}
$$

represent the spatial correlation of the $\mathbf{v}_m$, $\mathbf{v}_1$, and $\mathbf{v}_a$ vectors.

Note that the vector $\mathbf{v}_a$ is the replica associated with the directional cosine u_a, which determines where we're evaluating the beampattern.

MVDR beampattern for a single interferer (cont.)

Note that we can interpret the MVDR beampattern as:

$$B_{\mathrm{mvdr}}(u_a) = \frac{\Lambda_{\mathrm{mvdr}} N}{\sigma_w^2} \left[\underbrace{\rho_{ma}}_{\text{CBF: } \mathbf{v}_m} - \frac{N\sigma_I^2 \rho_{m1}}{1 + N\sigma_I^2} \underbrace{\rho_{1a}}_{\text{CBF: } \mathbf{v}_1} \right]$$

The ρ_{ma} and ρ_{1a} terms are equal to a conventional beamformer steered to $\mathbf{v}_m$ and $\mathbf{v}_1$, respectively.

Thus the overall beampattern is the difference between two scaled conventional patterns.

Additional insights from a look at value of BP at interferer location $\Rightarrow$

MVDR beampattern for a single interferer (cont.)

Consider the MVDR BP evaluated at the interferer location, $u_a = u_1$:

$$
\begin{aligned}
B_{\text{mvdr}}(u_1) &= \frac{\Lambda_{\text{mvdr}} N}{\sigma_w^2} \left[\rho_{m1} - \frac{N\sigma_I^2 \rho_{m1}}{1 + N\sigma_I^2} \underbrace{\rho_{11}}_{=1} \right] \\
&= \frac{\Lambda_{\text{mvdr}} N}{\sigma_w^2} \left[1 - \left(\frac{N\sigma_I^2}{1 + N\sigma_I^2} \right) \right] \rho_{m1}
\end{aligned}
$$

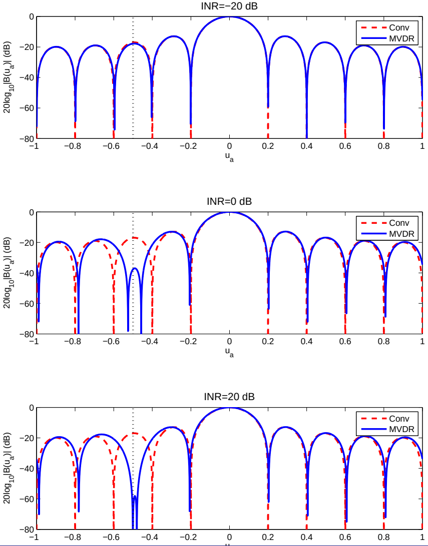
Thus the MVDR beamformer creates a partial null in the beampattern at the location of the interferer. Depth of this null depends on:

- $\sigma_I^2 =$ INR. As $\sigma_I^2 \to \infty$, the term in parentheses goes to 1, and an exact null is created in the beampattern.
- $\rho_{m1} =$ value of conventional BP at interferer location. If there is already a null at this location, then $\rho_{m1} = 0$.

Example of MVDR beampattern for 1 interferer

Plots show the MVDR and conventional beampatterns for a 10-element standard linear array at different INR's. Interferer is located at $u_1 = -0.5$, which is not a null in the conventional beampattern, *i.e.*, $\rho_{m1} \neq 0$.

- -20 dB INR: since σ_I^2 is small, $B_{\text{conv}} \approx B_{\text{mvdr}}$.
- 0 dB INR: σ_I^2 is high enough that $B_{\text{mvdr}}(u_1) < B_{\text{conv}}(u_1)$.
- +20 dB INR: deeper null created at $u_a = u_1$.



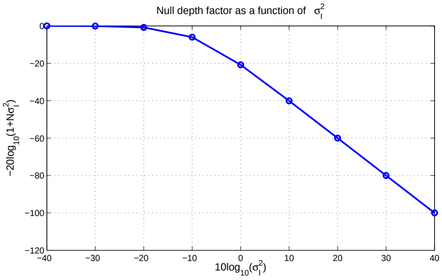
Depth of nulls in MVDR beampattern

The depth of the null in the MVDR beampattern is a function of

$$1 - \left(\frac{N\sigma_I^2}{1 + N\sigma_I^2} \right) = \frac{1}{1 + N\sigma_I^2}$$

The plot at right shows this factor (in dB) as a function of σ_I^2 in dB.

- Interferer will have almost no effect on the BP for $10\log_{10}(\sigma_I^2) \leq -20$ dB
- Interferer will drive BP down significantly for $10\log_{10}(\sigma_I^2) \geq 0$ dB

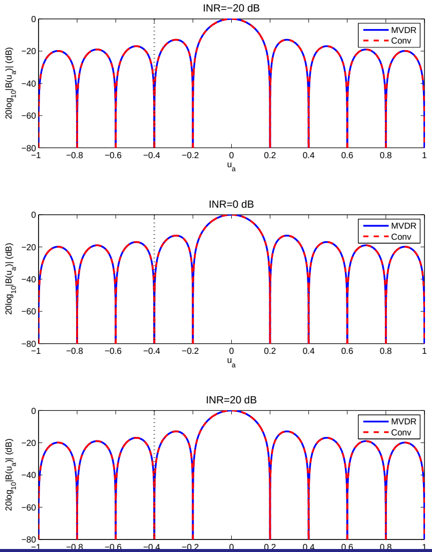


MVDR beampattern when interferer at null in B_{conv}

Plots illustrate what happens when the interferer is at the location of a null in the conventional pattern.

$\Rightarrow$ Regardless of INR level, the MVDR and conventional beampatterns are identical in this case.

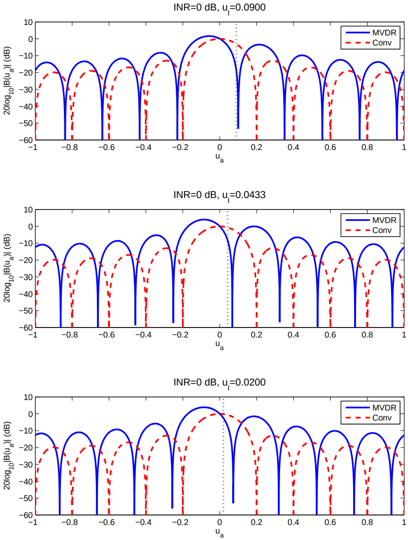
Plots show results for a 10-element standard linear array, but similar behavior would be seen for other arrays.



MVDR BP as fxn of interferer location: 0 dB INR

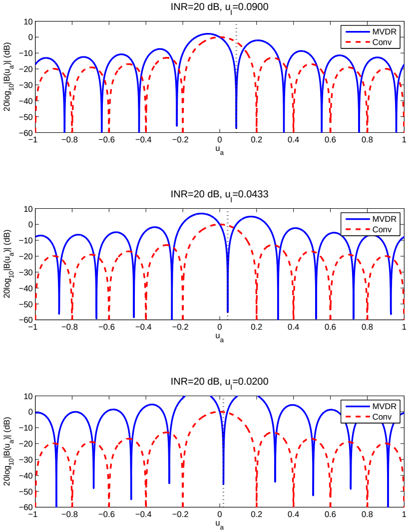
As discussed in class and explored in Problem Set 5, the behavior of the MVDR beampattern changes substantially as the interferer moves inside the mainlobe.

- Plots show results for $u_I = 0.09, 0.0433, 0.02$ and INR=0 dB.
- Unity gain maintained at $u_a = 0$.
- Mainlobe is distorted by presence of ML interferer.



MVDR BP as fxn of interferer location: 20 dB INR

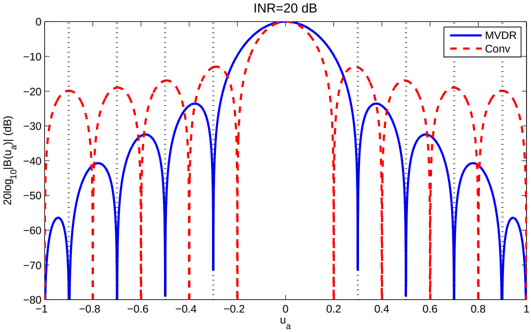
- Plots show results for $u_I = 0.09, 0.0433, 0.02$ and INR=0 dB.
- Unity gain still maintained at $u_a = 0$.
- Mainlobe is very distorted by presence of high INR ML interferer.
- In general ML interferers are very hard to deal with.



MVDR beampattern for multiple interferers

The following plot shows the MVDR and conventional BP for a 10-element standard linear array when the noise consists of 8 strong discrete interferers (INR=20 dB) outside the mainlobe plus spatially white noise.

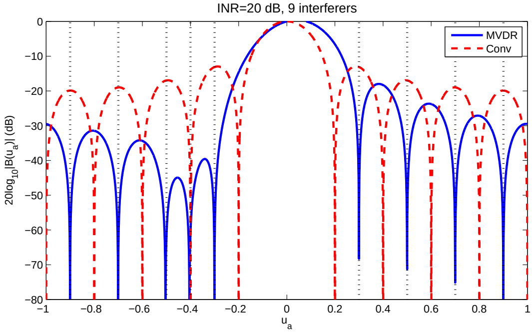
- MVDR processor has 8 deep nulls
- Unlike single-interferer case, here MVDR BF has wider mainlobe than conventional BF $\longrightarrow$ part of the price paid for nulls



MVDR beampattern for 9 interferers

The following plot shows that 9 equal-power interferers result in 9 deep nulls in the beampattern of the 10-element array

- Mainlobe is more distorted than in the 8-interferer case.
- Note: there are 5 interferers to the left of the mainlobe and 4 interferers to the right of the mainlobe.

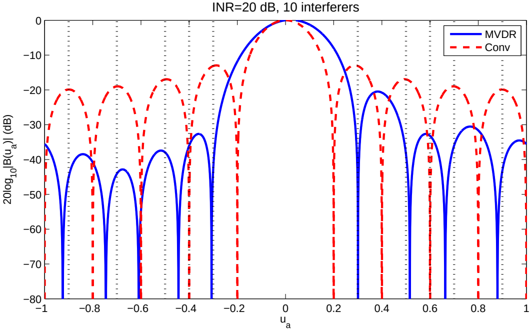


Can we null out 10 interferers with this array? ⇒

MVDR beampattern for 10 interferers

When there are 10 equal-power interferers incident on a 10-element array, the MVDR beamformer can no longer null all of them.

- 10-element array $\longrightarrow$ 10 coeff's in $\mathbf{w}_{\mathrm{mvdr}}$
- Do not have enough degrees of freedom to satisfy unity gain constraint *and* maintain 10 exact nulls with a 10-element array



Resulting MVDR BF may still be useful, but BP behavior is different when # of interferers is $\geq N$.

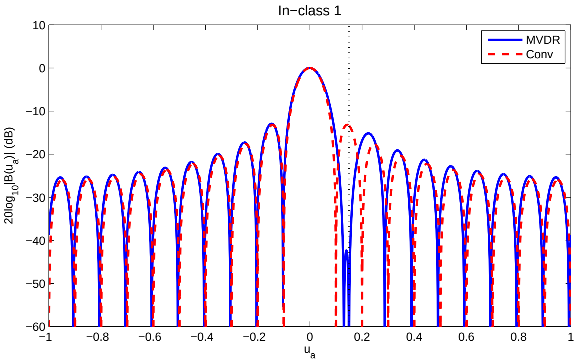
In-class problems

Consider a 20-element standard linear array. Can you predict how the MVDR beampattern will differ from the conventional beampattern in the following cases?

1. The noise field consists of spatially white noise plus one discrete interferer at $u_I = 0.15$ with INR=40 dB.
2. The noise field consists of spatially white noise plus two discrete interferers at $u_I = \pm 0.1$ with INR's of 20 dB.
3. The noise field consists of spatially white noise plus a discrete interferer at $u_I = -0.25$ with INR=20 dB and a discrete interferer at $u_I = .05$ with INR=0 dB.

In-class problem 1 solution

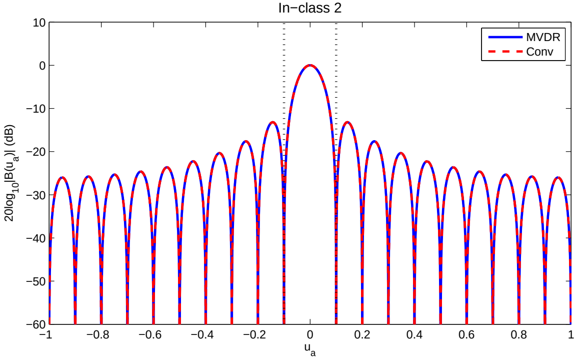
20-element standard linear array, $u_I = 0.15$, INR=40 dB



Since interferer outside mainlobe we expect to see a significant null at the interferer location in the MVDR pattern. There is no similar null in the conventional pattern.

In-class problem 2 solution

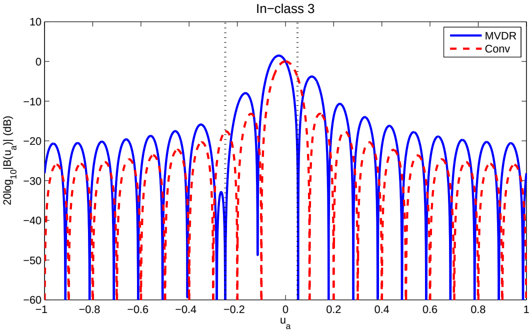
20-element standard linear array, $u_I = \pm 0.1$, INR=20 dB



Since interferers are at locations of nulls in conventional pattern, $B_{\mathrm{mvdr}} = B_{\mathrm{conv}}$ for this example.

In-class problem 3 solution

20-element standard linear array, $u_I = -0.25$/INR=20 dB and $u_I = 0.05$/INR=0 dB.



Expect to see significant null for interferer at $u_I = -0.25$ since it is outside mainlobe and has high INR. Since the interferer at $u_I = 0.05$ is inside the mainlobe, we expect some mainlobe distortion, though unity gain at broadside is maintained.

Summary: effect of interferers on MVDR beampattern

- If INR is low, MVDR beampattern resembles conventional beampattern.
- For modest INR ($\approx$ 0 dB), MVDR processor places partial nulls in beampattern at locations of interferers.
- For high INR, MVDR will place exact nulls at locations of interferers, assuming the number of discrete interferers is less than the number of elements.
- For interferers outside the mainlobe, the mainlobe of the MVDR processor is well-behaved (resembles conventional processor for $N >>$#interferers.
- For interferers inside the mainlobe, the mainlobe of the MVDR processor may be significantly distorted (esp. for high INR) as the processor tries to maintain the unity gain constraint and null out an interferer close to the desired signal direction.

Array gain

Array gain quantifies the improvement in signal-to-noise ratio due to using the array. It is defined as the ratio of the SNR at the output of the beamformer to the SNR at an individual input sensor:

$$\text{Array gain} = A = \frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{in}}}$$

Note that in general the noise includes white and colored noise, as well as planewave interferers. For that reason SNR is often called SINR, which stands for Signal-to-Interference-and-Noise Ratio.

How to compute these quantities? $\Rightarrow$

How to compute $\mathrm{SNR}_{\mathrm{in}}$

The diagonal elements of $\mathbf{S}_{\mathrm{sig}}$ and $\mathbf{S}_{\mathbf{n}}$ are the signal powers and noise powers, respectively.

Since the diagonal elements of $\mathbf{v}_s\mathbf{v}_s^{\mathrm{H}}$ are equal to 1, the signal power at each sensor is σ_s^2.

We obtain the average noise power by summing the diagonal elements of $\mathbf{S}_{\mathbf{n}}$ and dividing by the number of sensors:

$$\sigma_n^2 = \frac{\mathrm{tr}(\mathbf{S}_{\mathbf{n}})}{N}$$ where tr denotes the trace operation

Thus, the input SNR is:

$$\mathrm{SNR}_{\mathrm{in}} = \frac{\sigma_s^2}{\sigma_n^2}$$

How to compute $\mathrm{SNR}_{\mathrm{out}}$

If we process the snapshot using the weight vector **w**, the power in the signal component of the output is

$$P_{\mathrm{sig}} = \mathcal{E}\{|\mathbf{w}^{\mathrm{H}}\mathbf{x}_{\mathrm{sig}}|^2\} = \mathcal{E}\{\mathbf{w}^{\mathrm{H}}\mathbf{x}_{\mathrm{sig}}\mathbf{x}_{\mathrm{sig}}^{\mathrm{H}}\mathbf{w}^{\mathrm{H}}\} = \mathbf{w}^{\mathrm{H}}\mathbf{S}_{\mathrm{sig}}\mathbf{w} = \sigma_s^2|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2$$

Similarly, the noise power at the output of the beamformer is

$$P_{\mathrm{noise}} = \mathbf{w}^{\mathrm{H}}\mathbf{S}_{\mathbf{n}}\mathbf{w} = \sigma_n^2\mathbf{w}^{\mathrm{H}}\boldsymbol{\rho}_{\mathbf{n}}\mathbf{w},$$

where $\boldsymbol{\rho}_{\mathbf{n}}$ = normalized noise covariance matrix $\boldsymbol{\rho}_{\mathbf{n}} = \mathbf{S}_{\mathbf{n}}/\sigma_n^2$.
Thus the output SNR is:

$$\mathrm{SNR}_{\mathrm{out}} = \frac{\sigma_s^2|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2}{\sigma_n^2(\mathbf{w}^{\mathrm{H}}\boldsymbol{\rho}_{\mathbf{n}}\mathbf{w})}$$

Array gain ⇒

Formula for array gain

Thus the array gain for the BF with weight vector **w** is equal to:

$$A = \frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{in}}} = \frac{\sigma_s^2 |\mathbf{w}^{\mathrm{H}} \mathbf{v}_s|^2}{\sigma_n^2 (\mathbf{w}^{\mathrm{H}} \boldsymbol{\rho}_{\mathbf{n}} \mathbf{w})} \cdot \frac{\sigma_n^2}{\sigma_s^2} = \frac{|\mathbf{w}^{\mathrm{H}} \mathbf{v}_s|^2}{\mathbf{w}^{\mathrm{H}} \boldsymbol{\rho}_{\mathbf{n}} \mathbf{w}}$$

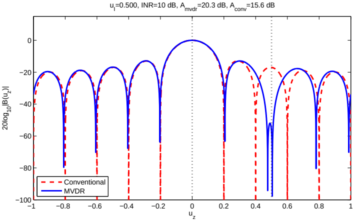
Note that if the weight vector is designed with a unity gain constraint for replica $\mathbf{v}_s$, then $|\mathbf{w}^{\mathrm{H}} \mathbf{v}_s| = 1$ and the array gain reduces to

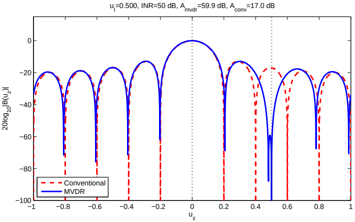
$$A = \frac{1}{\mathbf{w}^{\mathrm{H}} \boldsymbol{\rho}_{\mathbf{n}} \mathbf{w}}$$

Examples ⇒

Array gain example 1: interferer in sidelobes

The figures on the right show the MVDR and conventional beampatterns and array gains for a standard 10-element linear array with a single interferer. The top plot shows the results for 10 dB INR, and the bottom plot shows the results for 50 dB INR. The interferer is located at $u_I = 0.5$. The conventional beamformer obviously has lower array gain because the interferer is located near one of the sidelobe peaks, whereas the MVDR beamformer has a partial null at the inteferer location.

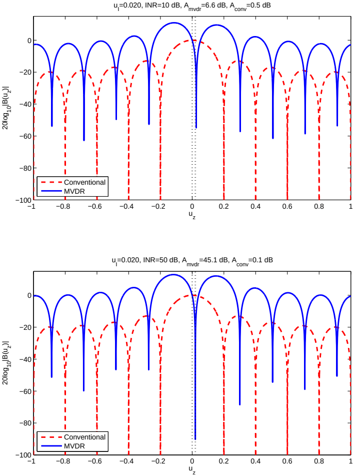




What happens when interferer is in the mainlobe? $\Rightarrow$

Array gain example 2: interferer in mainlobe

MVDR and conventional beampatterns and array gains for a standard 10-element linear array with a single interferer in the mainlobe. Top plot shows results for 10 dB INR, and bottom plot shows results for 50 dB INR. The conventional beamformer has very low array gain because the interferer is very close to the steering direction. The MVDR beamformer is able to put a null at the interferer location, so its array gain is relatively high, even though the beampattern is quite distorted. As we see on the following slides, the white noise gain is seriously affected with an interferer in the mainlobe.



White noise gain

White noise gain is the array gain when $\mathbf{S_n} = \sigma_w^2 \mathbf{I}$ (and $\boldsymbol{\rho}_\mathbf{n} = \mathbf{I}$).

$$A_w = \frac{|\mathbf{w}^{\mathrm{H}} \mathbf{v}_s|^2}{\mathbf{w}^{\mathrm{H}} \boldsymbol{\rho}_\mathbf{n} \mathbf{w}} = \frac{|\mathbf{w}^{\mathrm{H}} \mathbf{v}_s|^2}{\mathbf{w}^{\mathrm{H}} \mathbf{w}}$$

If $\mathbf{w}$ is constrained to give unit gain in the look direction, $|\mathbf{w}^{\mathrm{H}} \mathbf{v}_s| = 1$:

$$A_w = \frac{1}{\mathbf{w}^{\mathrm{H}} \mathbf{w}} = \frac{1}{T_{se}}$$

White noise gain is the inverse of the 2-norm-squared of the weight vector. Recall from Ch. 2 of Van Trees that white noise gain is the inverse of the sensitivity function T_{se}.

- Thus, the lower the white noise gain, the higher the sensitivity of the beamformer to mismatch.

White noise gain continued . . .

Note that the weight vector used in the white noise gain calculation may be computed using a non-white noise covariance, *e.g.*,

$$\mathbf{w} = \mathbf{w}_{\mathrm{mvdr}} = \Lambda_{\mathrm{mvdr}} \mathbf{S_n} \mathbf{v}_m$$

In this case $\mathbf{S_n}$ may contain white noise plus colored noise plus one or more planewave interferers. The white noise gain is still

$$A_{w-\mathrm{mvdr}} = \frac{1}{\mathbf{w}_{\mathrm{mvdr}}^{\mathrm{H}} \mathbf{w}_{\mathrm{mvdr}}}$$

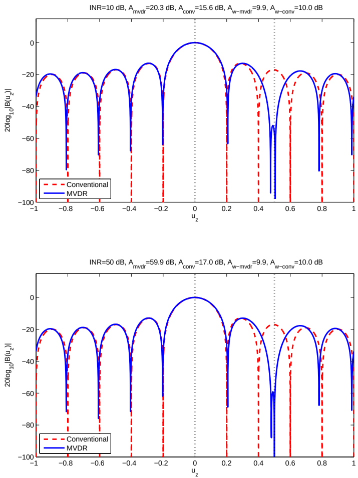
Examples ⇒

White noise gain example 1: interferer in sidelobes

Beampatterns for a standard 10-element array with interferer in the sidelobes at 10 dB or 50 dB INR.

	10 dB INR	50 dB INR
A_{mvdr}	20.3 dB	59.9 dB
A_{conv}	15.6 dB	17.0 dB
$A_{\mathrm{w-mvdr}}$	9.9 dB	9.9 dB
$A_{\mathrm{w-conv}}$	10.0 dB	10.0 dB

With interferer in the sidelobes the white noise gain for the MVDR and conventional processors are approximately equal.

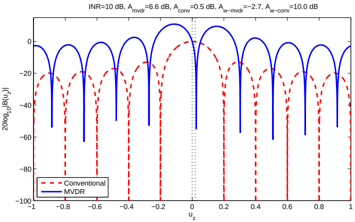


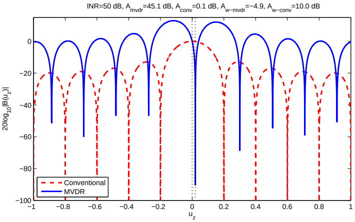
White noise gain example 2: interferer in mainlobe

Beampatterns for a standard 10-element array with interferer in the mainlobe at 10 dB or 50 dB INR.

	10 dB INR	50 dB INR
A_{mvdr}	6.6 dB	45.1 dB
A_{conv}	0.5 dB	0.1 dB
$A_{\mathrm{w-mvdr}}$	-2.7 dB	-4.9 dB
$A_{\mathrm{w-conv}}$	10.0 dB	10.0 dB

With interferer in the mainlobe the white noise gain for the MVDR processor is substantially decreased. The conventional processor is unaffected because it does not attempt to null the strong mainlobe interferer.

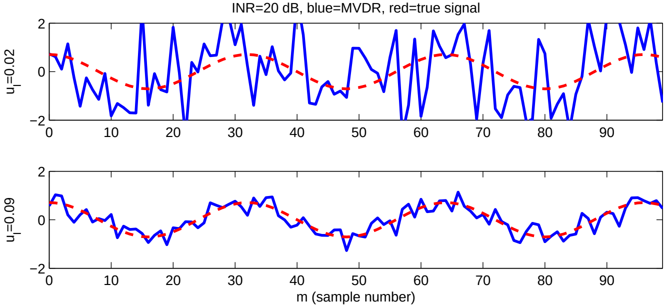




White noise gain example 3: signals

The following plots illustrate the effect of white noise gain on the output signals of a beamformer. The top plot shows the output (real part only) of an MVDR beamformer with white noise gain of -4.7 dB and the bottom plot shows the output of an MVDR beamformer with white noise gain 7.1 dB.

The lower A_w is, the more white noise comes through, making it harder to distinguish the underlying signal (shown in red).



Note that the difference in white noise gain in these two plots is caused by the interferer moving closer to the desired signal in the mainlobe.

In-class problems

4. Suppose that you have a 30-element linear array. What is the maximum possible value for the white noise gain for this array? What beamformer is guaranteed to achieve this white noise gain?
5. Suppose that you use two conventional beamformers to process data. The data contains the desired planewave signal plus spatially white noise. Beamformer #1 has a white noise gain of 10 dB and Beamformer #2 has a white noise gain of 20 dB. How do you expect the two outputs to differ? How do you build conventional beamformers with different white noise gains?

In-class problem 4 solution

Consider deriving a beamformer by maximizing white noise gain given a unity gain constraint, *i.e.*,

$$\max \left\{ \frac{1}{\mathbf{w}^{\mathrm{H}}\mathbf{w}} \right\} \text{ subject to } \mathbf{w}^{\mathrm{H}}\mathbf{v}_m = 1$$

This problem is equivalent to minimizing $\mathbf{w}^{\mathrm{H}}\mathbf{w}$, *i.e.*,

$$\min \left\{ \mathbf{w}^{\mathrm{H}}\mathbf{w} \right\} \text{ subject to } \mathbf{w}^{\mathrm{H}}\mathbf{v}_m = 1$$

Solution via LaGrange multipliers:

$$Q = \mathbf{w}^{\mathrm{H}}\mathbf{w} + \lambda(\mathbf{w}^{\mathrm{H}}\mathbf{v}_m - 1)$$

$$\nabla_{\mathbf{w}^{\mathrm{H}}} Q = \mathbf{w} + \lambda\mathbf{v}_m = 0 \qquad \nabla_{\lambda} Q = \mathbf{w}^{\mathrm{H}}\mathbf{v}_m - 1 = 0$$

In-class problem 4 solution continued ...

Setting the gradients equal to zero and solving yields:

$$\nabla_{\mathbf{w}^{\mathrm{H}}} Q = \mathbf{w} + \lambda \mathbf{v}_m = 0$$

$$\mathbf{w} = -\lambda \mathbf{v}_m \tag{1}$$

and

$$\nabla_{\lambda} Q = \mathbf{w}^{\mathrm{H}} \mathbf{v}_m - 1 = 0$$

$$\mathbf{w}^{\mathrm{H}} \mathbf{v}_m = 1 \tag{2}$$

Substituting 2 into 1 and solving yields:

$$\lambda = -\frac{1}{\mathbf{v}_m^{\mathrm{H}} \mathbf{v}_m}$$

Thus

$$\mathbf{w} = \frac{1}{\mathbf{v}_m^{\mathrm{H}} \mathbf{v}_m} \mathbf{v}_m = \frac{\mathbf{v}_m}{N}$$

In-class problem 4 solution continued . . .

Thus the beamformer that maximizes white noise gain is the conventional beamformer.

The maximum white noise gain is

$$A_{\mathrm{w-max}} = \frac{1}{\frac{\mathbf{v}_m^{\mathrm{H}}}{N}\frac{\mathbf{v}_m}{N}} = \frac{N^2}{\mathbf{v}_m^{\mathrm{H}}\mathbf{v}_m} = \frac{N^2}{N} = N$$

(The Schwartz inequality can also be used to show that this is the maximum white noise gain.)

For a 30-element array, the maximum white noise gain in dB is $10\log_{10}|30| = 14.8$ dB. This white noise gain is achieved by the conventional beamformer with uniform weighting.

In-class problem 5

Suppose that you use two conventional beamformers to process data. The data contains the desired planewave signal plus spatially white noise. Beamformer #1 has a white noise gain of 10 dB and Beamformer #2 has a white noise gain of 20 dB. How do you expect the two outputs to differ?

- Will the true signal coming through each beamformer differ in amplitude? *Answer:* No. Both beamformers are designed with a unity gain constraint.
- How will the noise coming through each beamformer differ? *Answer:* The power of the noise at the output of Beamformer #1 will be higher (by 10 dB) than the power of the noise at the output of Beamformer #2.

In-class problem 5 continued

How do you build conventional beamformers with different white noise gains?

Answer: recall that the white noise gain of a conventional beamformer is equal to $10 \log_{10} |N|$ in dB. To change the white noise gain of this beamformer, you must change the number of sensors. In this case Beamformer #1 must have 10 sensors since it has a white noise gain of 10 dB. Beamformer #2 must have 20 sensors since it has a white noise gain of 20 dB.

Summary: array gain

- Array gain quantifies the improvement in signal-to-noise ratio between the input (*i.e.,* a single sensor) and the output of an array.
- White noise gain measures the improvment in SNR between the array's input and output when the input noise is assumed to be spatially white.
- The white noise gain is an indicator of the sensitivity of the beamformer to mismatch. The lower the white noise gain, the higher the sensitivity.
- The white noise gain of the MVDR beamformer is substantially decreased when the interferer moves inside the mainlobe. The white noise gain of the conventional beamformer is constant regardless of the interferer location since it's beampattern is fixed.

Summary: array gain continued . . .

- The array gain of the MVDR beamformer is always greater than or equal to that of the conventional beamformer:

$$A_{\text{mvdr}} \geq A_{\text{conv}}$$

The equality holds when the noise is spatially white, *i.e.*, $\rho_{\mathbf{n}} = \mathbf{I}$.

- The white noise gain of the conventional beamformer is always greater than or equal to the white noise gain of the MVDR beamformer:

$$A_{\text{w-conv}} = N \geq A_{\text{w-mvdr}}$$

The maximum value of white noise gain is equal to the number of sensors in the array.