Determination of a Minimal Realization Using Kalman Canonical Forms

A. System Model

The system to be considered consists of two second-order subsystems connected in series. The simulation diagram for the system is shown in Fig. 1. Based on the state variables shown in the figure, the \( \{A, B, C, D\} \) matrices are:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
2.5 & 2.5 & -1 & 0 \\
2.5 & 2.5 & 0 & -3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0 & 1 & 1
\end{bmatrix}, \quad D = 0
\]

(1)

The eigenvalues of \( A \) are \( \lambda = \{0; -1; -2; -3\} \). The controllability and observability matrices are given by

\[
P = \begin{bmatrix}
B & AB & A^2B & A^3B
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & -2 & 4 & -8 \\
0 & 5 & -10 & 20 \\
0 & 5 & -20 & 70
\end{bmatrix}
\]

(2)

\[
Q = \begin{bmatrix}
C \\
CA \\
CA^2 \\
CA^3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 1 \\
5 & 5 & -1 & -3 \\
-10 & -20 & 1 & 9 \\
25 & 65 & -1 & -27
\end{bmatrix}
\]

(3)

The rank of the controllability matrix is \( r_P = 3 \) and the rank of the observability matrix is \( r_Q = 3 \). Since \( r_P < n = 4 \) and \( r_Q < n \), the system is neither completely controllable nor completely observable. The goal of this example is to determine an irreducible realization (minimal realization) for this system, that is, a state space model with the fewest number of state variables that will exactly reproduce the input/output characteristics from \( u(t) \) to \( y(t) \) in the time domain or from \( U(s) \) to \( Y(s) \) in the complex frequency domain. The Kalman controllable canonical and Kalman observable canonical forms will be used to accomplish this. The QR decomposition (MATLAB function \( qr \)) will be used to determine the orthonormal basis vectors for the subspaces.

Since the given system is neither completely controllable nor completely observable, it may be necessary to obtain both of the Kalman canonical forms. Since the ranks of \( P \) and \( Q \) are equal, it makes no real difference which decomposition we do first. If one of the ranks was smaller than the other, the Kalman decomposition for the form with the smaller rank would be done first. In this example, the Kalman controllable canonical form will be obtained first.

Fig. 1. Simulation diagram for the series connection of two systems.
B. Kalman Controllable Canonical Form

The Kalman canonical form that decomposes state space \( \Sigma \) into controllable and uncontrollable subspaces yields the following state equations in the new basis vectors.

\[
\begin{bmatrix}
\dot{w}_1 \\
\vdots \\
w_2
\end{bmatrix} = \begin{bmatrix} T_1^T AT_1 & T_1^T AT_2 \end{bmatrix} \begin{bmatrix} w_1 \\
\vdots \\
w_2
\end{bmatrix} + \begin{bmatrix} T_1^T B \end{bmatrix} u, \quad y = \begin{bmatrix} CT_1 & CT_2 \end{bmatrix} \begin{bmatrix} w_1 \\
\vdots \\
w_2
\end{bmatrix} + Du
\]

(4)

where \( n \times n \) matrix \( T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \), and the \( r_P \) columns of \( T_1 \) span the controllable subspace \( \Sigma_c \). The dimension of \( \Sigma_c \) and the dimension of the state vector \( w_1 \) is \( r_P = 3 \). The orthonormal basis vectors for the controllable and uncontrollable subspaces are:

\[
T = \begin{bmatrix}
-0.7071 & 0.1387 & -0.1334 & \cdots & -0.6804 \\
-0.7071 & -0.1387 & 0.1334 & \cdots & 0.6804 \\
0 & 0.6934 & -0.6672 & \cdots & 0.2722 \\
0 & 0.6934 & 0.7206 & \cdots & 0
\end{bmatrix}
\]

(5)

Since we are interested in a minimal order system that will be completely controllable and completely observable, we do not need to keep the uncontrollable states in the model. Therefore, vector \( w_2 \) and everything associated with it will be ignored. The new system model is

\[
\dot{w}_1 = (T_1^T AT_1) w_1 + (T_1^T B) u = A_c w_1 + B_c u, \quad y = (CT_1) w_1 = C_c w_1
\]

(6)

The \( D \) matrix is omitted from Eqn. (6) since \( D = 0 \) in all realizations of the system. The matrices for this new system model are

\[
A_c = \begin{bmatrix}
-1 & -0.1961 & 0.1887 \\
-5.099 & -1.9615 & -0.9993 \\
0 & -0.9993 & 2.0385
\end{bmatrix}, \quad B_c = \begin{bmatrix}
-1.4142 \\
0 \\
0
\end{bmatrix}, \quad C_c = \begin{bmatrix}
0 & 1.3868 & 0.0534
\end{bmatrix}
\]

(7)

This third-order system is completely controllable because of the way it was constructed from the QR decomposition. The eigenvalues of \( A_c \) are \( \lambda_c = \{0; -2; -3\} \). However, it is not guaranteed to be observable. Controllability will be verified for completeness and observability will be checked.

\[
P_c = \begin{bmatrix} B_c & A_c B_c & A_c^2 B_c \end{bmatrix} = \begin{bmatrix}
-1.4142 & 1.4142 & -2.8284 \\
0 & 7.2111 & -21.356 \\
0 & 0 & -7.2058
\end{bmatrix}, \quad \text{Rank}(P_c) = r_{Pc} = 3 = r_P
\]

(8)

\[
Q_c = \begin{bmatrix} C_c & C_c A_c & C_c^2 A_c \\
C_c A_c & C_c A_c^2 & C_c A_c^3 \end{bmatrix} = \begin{bmatrix}
0 & 1.3868 & 0.0534 \\
-7.0711 & -2.7735 & -1.4945 \\
21.2132 & 8.3205 & 4.4836
\end{bmatrix}, \quad \text{Rank}(Q_c) = r_{Qc} = 2 < r_P
\]

(9)

The rank of the controllability matrix is indeed equal to 3, as it must be. Therefore, this third-order system is completely controllable. However, it is not completely observable since the rank of \( Q_c = 2 < r_P \). Because of this, the Kalman observable form will have to be obtained, using this third-order system represented by \( \{A_c, B_c, C_c\} \) as the starting point.

C. Kalman Observable Canonical Form

The Kalman canonical form that decomposes the controllable state space \( \Sigma_c \) into observable and unobservable subspaces yields the following state equations in the new basis vectors.

\[
\begin{bmatrix}
\dot{v}_1 \\
\vdots \\
v_2
\end{bmatrix} = \begin{bmatrix} V_1^T A_c V_1 & 0 \\
\vdots & \vdots \\
V_2^T A_c V_1 & V_2^T A_c V_2
\end{bmatrix} \begin{bmatrix} v_1 \\
\vdots \\
v_2
\end{bmatrix} + \begin{bmatrix} V_1^T B_c \\
\vdots \\
V_2^T B_c
\end{bmatrix} u, \quad y = \begin{bmatrix} C_c V_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\
v_2
\end{bmatrix}
\]

(10)
where \( r_P \times r_P \) matrix \( V = \begin{bmatrix} V_1 & : & V_2 \end{bmatrix} \), and the \( r_Q \) columns of \( V_1 \) span the observable subspace \( \Sigma_o \). The dimension of \( \Sigma_o \) and the dimension of the state vector \( v_1 \) is \( r_Q = 2 \). The orthonormal basis vectors for the observable and unobservable subspaces are:

\[
V = \begin{bmatrix} 0 & 0.9813 & : & -0.1925 \\ -0.9939 & -0.0074 & : & -0.0377 \\ -0.0385 & 0.1923 & : & 0.9806 \end{bmatrix}
\] (11)

As before, since we are interested in a minimal order system that will be completely controllable and completely observable, we do not need to keep the unobservable states in the model. Therefore, vector \( v_2 \) and everything associated with it will be ignored. The new system model is

\[
\dot{v}_1 = (V_1^T A_c V_1) v_1 + (V_1^T B_c) u = A_{co} v_1 + B_{co} u, \quad y = (C_c V_1) v_1 = C_{co} v_1
\] (12)

The matrices for this new system model are

\[
A_{co} = \begin{bmatrix} -2.0385 & 5.1923 \\ 0.3775 & -0.9615 \end{bmatrix}, \quad B_{co} = \begin{bmatrix} 0 \\ -1.3878 \end{bmatrix}, \quad C_{co} = \begin{bmatrix} -1.3878 & 0 \end{bmatrix}
\] (13)

This second-order system is completely observable because of the way it was constructed from the QR decomposition. It is also guaranteed to be controllable since the \( \{A_c, B_c, C_c\} \) model “exists” only in the controllable subspace. The eigenvalues of \( A_{co} \) are \( \lambda_{co} = \{0; -3\} \). Controllability and observability will be verified for completeness.

\[
P_{co} = \begin{bmatrix} B_{co} & A_{co} B_{co} \end{bmatrix} = \begin{bmatrix} 0 & -7.2058 \\ -1.3878 & 1.3344 \end{bmatrix}, \quad \text{Rank} \left( P_{co} \right) = \text{Rank} \left( B_{co} \right) = 2 = r_Q
\] (14)

\[
Q_{co} = \begin{bmatrix} C_{co} & C_{co} A_{co} \end{bmatrix} = \begin{bmatrix} -1.3868 & 0 \\ 2.8289 & -7.2058 \end{bmatrix}, \quad \text{Rank} \left( Q_{co} \right) = \text{Rank} \left( C_{co} \right) = 2 = r_Q
\] (15)

Since this system given by \( \{A_{co}, B_{co}, C_{co}\} \) is both completely controllable and completely observable, it is irreducible. It is the minimum order of any state space system that will exactly give the same relationship between the input and the output as the original system. Since there were four states in the original model and only two in this minimal realization, two of those original states were not needed in terms of the input/output relationship. Since the states for this system are completely controllable and completely observable, the state space for this minimal realization is the intersection of the controllable and observable subspaces of the original state space \( \Sigma \), that is, \( \Sigma_{co} = \Sigma_c \cap \Sigma_o \).

D. Transfer Functions for the System

The transfer function for the two subsystems are

\[
H_1(s) = \frac{2(s+1)}{s(s+2)} = \frac{1}{s} + \frac{1}{s+2}, \quad H_2(s) = \frac{5(s+2)}{(s+1)(s+3)} = \frac{2.5}{s+1} + \frac{2.5}{s+3}
\] (16)

It is clear from (16) that there are two pole-zero cancellations between \( H_1(s) \) and \( H_2(s) \). The loss of controllability and observability is due to these cancellations. Specifically, the eigenvalue at \(-1\) from the \((s+1)\) term in the denominator of \( H_2(s) \) is uncontrollable because of the zero in \( H_1(s) \) at the same location. Nothing that is done with \( u(t) \) can affect that eigenvalue. The eigenvalue at \(-2\) from the \((s+2)\) term in the denominator of \( H_1(s) \) is unobservable because of the zero in \( H_2(s) \). No response by that eigenvalue will be seen at the output \( y(t) \).

The transfer function for the minimal realization is

\[
H_{\min}(s) = C_{co} (sI - A_{co})^{-1} B_{co} = \frac{10}{s(s+3)}
\] (17)

\( H_{\min}(s) = H_1(s)H_2(s) \) after both pole-zero cancellations are performed.

For single-input, single-output (SISO) system, any time there is a pole-zero cancellation in the transfer function, any state space realization generated from the complete transfer function (without doing the cancellation) will suffer either a loss of controllability or a loss of observability. Which property is lost depends on the state space realization chosen. In this example, there were two possible pole-zero cancellations, positioned in the two subsystems such that both properties of controllability and observability were lost.