Problem 1a.
The system is completely controllable, and the performance index is completely observable. Therefore, the optimal control exists and is bounded, the closed-loop system is asymptotically stable, the state trajectory is bounded and goes to the origin, and the performance index is bounded.

Problem 1b.
The system is completely controllable, but the performance index is not completely observable. Since the state variable that is not observable is unstable, the performance index is not detectable. The optimal control exists and is bounded, and the performance index is bounded. However, the closed-loop system is unstable, and state variable $x_2$ is unbounded.

Problem 1c.
The system is completely controllable, but the performance index is not completely observable. Since the state variable that is not observable is asymptotically stable, the performance index is detectable. The optimal control exists and is bounded, and the performance index is bounded. The closed-loop system is stable, and the state trajectory is bounded.

Problem 1d.
The performance index is completely observable, so all state variables affect the value of $J$. The system is not completely controllable, and since the uncontrollable state is unstable, the system is not detectable. Therefore, the control is unbounded, the performance index is unbounded, the closed-loop system is unstable, and the state trajectory is unbounded.

Problem 2.
The performance index and the two constraint equations are

$$J = 0.5 \left( (x_1 - x_2)^2 + (y_1 - y_2)^2 \right)$$

$$\quad \quad \quad \quad (x_1 + 1)^2 + y_1^2 - 4 = 0, \quad y_2 - x_2 + 5 = 0$$

The modified performance index—includes the constraint equations—is

$$J_1 = 0.5 \left( (x_1 - x_2)^2 + (y_1 - y_2)^2 \right) + \lambda_1 \left( (x_1 + 1)^2 + y_1^2 - 4 \right) + \lambda_2 (y_2 - x_2 + 5)$$

The first derivatives of $J_1$ with respect to all the variables ($x_1, y_1, x_2, y_2, \lambda_1, \lambda_2$) are

$$\frac{\partial J_1}{\partial x_1} = 0 = (x_1 - x_2) + 2\lambda_1 (x_1 + 1), \quad \frac{\partial J_1}{\partial y_1} = 0 = (y_1 - y_2) + 2\lambda_1 y_1$$

$$\frac{\partial J_1}{\partial x_2} = 0 = -(x_1 - x_2) - \lambda_2, \quad \frac{\partial J_1}{\partial y_2} = 0 = -(y_1 - y_2) + \lambda_2$$

$$\frac{\partial J_1}{\partial \lambda_1} = 0 = (x_1 + 1)^2 + y_1^2 - 4, \quad \frac{\partial J_1}{\partial \lambda_2} = 0 = y_2 - x_2 + 5$$

From the first expressions in (3) and (4),

$$x_2 = x_1 + 2\lambda_1 (x_1 + 1) = x_1 + \lambda_2 \Rightarrow \lambda_2 = 2\lambda_1 (x_1 + 1)$$

From the second expressions in (3) and (4),

$$y_2 = y_1 + 2\lambda_1 y_1 = y_1 - \lambda_2 \Rightarrow \lambda_2 = -2\lambda_1 y_1$$

From (6) and (7),

$$y_1 = -(x_1 + 1)$$

From (8) and the circle constraint equation in (1),

$$x_1^2 + 2x_1 - 1 = 0 \Rightarrow x_1 = -2.4142 \text{ or } + 0.4142$$
From the geometry of the problem, it is obvious that $x_1 = 0.4142$, and from (8) $y_1 = -1.4142$.
From (4) and (8),

$$\lambda_2 = (y_1 - y_2) = -(x_1 - x_2) \Rightarrow x_2 - x_1 = -x_1 - 1 - y_2 \Rightarrow x_2 = -(y_2 + 1) \quad (10)$$

Using (10) and the straight line constraint equation from (1), $x_2 = 2$, $y_2 = -3$.
The minimum distance between the circle and the straight line is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 2.2426 \quad (11)$$

**Problem 3a.**
The ARE can be solved with $q_{11}$ left in symbolic form. Substituting the given matrices into the ARE yields after the tedious algebra

$$S = A^T \left[ S - SB \left( B^T SB + R \right)^{-1} B^T \right] S A + Q \quad (12)$$
\[
\begin{bmatrix}
s_{11} & s_{12} \\
s_{12} & s_{22}
\end{bmatrix} = \begin{bmatrix}
q_{11} + \frac{4s_{11}}{s_{11} + 1} & \frac{1.2s_{12}}{s_{11} + 1} \\
\frac{1.2s_{12}}{s_{11} + 1} & 1 + \frac{0.36(s_{11}s_{22} + s_{22} - s_{12}^2)}{s_{11} + 1}
\end{bmatrix}
\]  

(13)

Equating the two sides of the equation element-by-element gives us the following three expressions:

\[
s_{11} = q_{11} + \frac{4s_{11}}{s_{11} + 1} \Rightarrow s_{11} - (3 + q_{11})s_{11} = 0
\]

\[
\Rightarrow s_{11} = \frac{(3 + q_{11}) + \sqrt{(3 + q_{11})^2 + 4q_{11}}}{2}
\]

\[
s_{12} = \frac{1.2s_{12}}{s_{11} + 1} \Rightarrow s_{12} (s_{11} - 0.2) = 0 \Rightarrow s_{12} = 0
\]

\[
s_{22} = 1 + \frac{0.36(s_{11}s_{22} + s_{22} - s_{12}^2)}{s_{11} + 1} \Rightarrow 0.64s_{22} (s_{11} + 1) = (s_{11} + 1)
\]

\[
\Rightarrow s_{22} = 1/0.64 = 1.5625
\]

Therefore, only \(s_{11}\) depends on the value of \(q_{11}\); the other elements are constant.

The optimal gain becomes

\[
G = (B^TSB + R)^{-1}B^TSA = \begin{bmatrix}
\frac{2s_{11}}{s_{11} + 1} & 0
\end{bmatrix}
\]

(17)

The closed-loop system matrix is

\[
A_{CL} = A - BG = \begin{bmatrix}
\frac{2}{s_{11} + 1} & 0 \\
0 & 0.6
\end{bmatrix}
\]

(18)

so one of the closed-loop eigenvalues is \(\lambda_{CL} = 0.6\), and the other is

\[
\lambda_{CL} = \frac{2}{s_{11} + 1} = \frac{4}{(5 + q_{11}) + \sqrt{(3 + q_{11})^2 + 4q_{11}}}
\]

(19)

**Problem 3b.**

For \(q_{11} = +2.5\),

\[
S = \begin{bmatrix}
5.9221 & 0 \\
0 & 1.5625
\end{bmatrix}, \quad G = \begin{bmatrix}
1.7111 & 0
\end{bmatrix}, \quad \lambda_{CL} = \begin{bmatrix}
0.2889 \\
0.6
\end{bmatrix}
\]

(20)

For \(q_{11} = -2.5\),

\[
S = \begin{bmatrix}
0.25 + 3.1225j & 0 \\
0 & 1.5625
\end{bmatrix}, \quad G = \begin{bmatrix}
1.779 + j0.552 & 0
\end{bmatrix}, \quad \lambda_{CL} = \begin{bmatrix}
0.221 - j0.552 \\
0.6
\end{bmatrix}
\]

(21)

With this \(q_{11}\) the gain is complex, so the state and control trajectories will no longer be real numbers—that does not correspond to a real system—so this weighting matrix would not be valid. The assumption of \(Q > 0\) is violated in this case.

As \(q_{11} > 0\) increases in value, the closed-loop eigenvalue that depends on \(q_{11}\) decreases in value; it moves closer to the origin of the \(z\)-plane, improving the relative stability of the system. The other closed-loop eigenvalue is fixed at 0.6, the location of one of the open-loop eigenvalues.

**Problem 3c.**

With \(u_k = -Gx_k\), the state and control trajectories are:
\[
H_k = Qx_k + Ru_k + \lambda_{k+1}^T f^k(x_k, u_k), \quad J_1 = Sx_N + \sum_{k=0}^{N-1} (H_k - \lambda_{k+1}^T x_{k+1})
\] (22)

Taking the first-order variations in \( J_1 \)
\[
dJ_1 = Sdx_N + \sum_{k=0}^{N-1} \left[ \left( \frac{\partial H_k}{\partial x_k} \right)^T dx_k + \left( \frac{\partial H_k}{\partial u_k} \right)^T du_k + \left( \frac{\partial H_k}{\partial \lambda_{k+1}} \right)^T d\lambda_{k+1} - \lambda_{k+1}^T dx_{k+1} - x_{k+1}^T d\lambda_{k+1} \right]
\] (23)

\[
dJ_1 = Sdx_N + \sum_{k=0}^{N-1} \left[ \left( \frac{\partial H_k}{\partial x_k} \right)^T dx_k + \left( \frac{\partial H_k}{\partial u_k} \right)^T du_k + \left( \frac{\partial H_k}{\partial \lambda_{k+1}} \right)^T d\lambda_{k+1} - \lambda_{k+1}^T dx_{k+1} - x_{k+1}^T d\lambda_{k+1} \right]
\] (24)

The boundary conditions are
\[
dx_0 = 0, \quad \lambda_N = S^T \] (25)

The other necessary conditions are
\[
\frac{\partial H_k}{\partial x_k} = \lambda_k = Q^T + \left[ \frac{\partial f^k(x_k, u_k)}{\partial x_k} \right] \lambda_{k+1} \] (26)
\[
\frac{\partial H_k}{\partial u_k} = 0 \Rightarrow R^T = -\left[ \frac{\partial f^k(x_k, u_k)}{\partial u_k} \right] \lambda_{k+1} \] (27)
\[
\frac{\partial H_k}{\partial \lambda_{k+1}} = x_{k+1} = f^k(x_k, u_k) \] (28)

The stationarity equation (27) does not have \( u_k \) appearing explicitly. If the state equations are linear in the control, then \( u_k \) does not appear at all in (27). Then there is no way to establish a relationship between \( u_k \) and \( \lambda_{k+1} \). If the state equations are also linear in the state, then \( x_k \) does not appear in (26).
Fig. 2. Optimal state and control trajectories for Problem #3 with $q_{11} = 2.5$. 