Minimum Fuel Optimal Control Example
For A Scalar System

A. Problem Statement

This example illustrates the minimum fuel optimal control problem for a particular first-order (scalar) system. The derivation of the general solution to this problem is found in the course textbook1. The expressions for the switching curve, the switching time, and the minimum time will be developed, and several examples with different initial states and different target states will be presented.

The general form for the system model, control constraint, and minimum fuel performance index are

$$\dot{x}(t) = f[x(t), t] + B[x(t), t]u(t), \quad |u_i(t)| \leq 1, \forall i \in [1, m], \quad J = \int_0^T e^T |u(t)| \, dt \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The specific expressions for the first-order system for this example, with $n = m = 1$, are given by

$$\dot{x}(t) = -x(t) + u(t), \quad |u(t)| \leq 1, \quad J = \int_0^T |u(t)| \, dt \quad (2)$$

For this system the terms in the general state space model and performance index have the following values: $f[x(t), t] = -1$, $B[x(t), t] = 1$, and $e^T = 1$. The Hamiltonian for the system is

$$H(t) = |u(t)| - \lambda(t)x(t) + \lambda(t)u(t) \quad (3)$$

where $\lambda(t)$ is the scalar costate variable (Lagrange multiplier) for this system. The solution to the minimum fuel problem is the control signal $u(t)$ that satisfies the state and costate equations and that minimizes the Hamiltonian, such that

$$H[x^*, \lambda^*, u^*, t] \leq H[x^*, \lambda^*, u, t] \quad \forall \text{ admissible } u(t) \quad (4)$$

which for this problem becomes

$$\text{Minimize } |u(t)| + \lambda(t)u(t), \quad |u(t)| \leq 1 \quad (5)$$

The solution to this minimization problem follows the form for the general case, and is

$$u^*(t) = -dez \left[ \frac{k^T \lambda(t)}{c_i} \right] = -dez[\lambda(t)] \quad (6)$$

where $dez$ is the deadzone function defined by

$$dez = \begin{cases} 
OUT = 1, & \text{when } IN > 1 \\
OUT = -1 & \text{when } IN < -1 \\
OUT = 0 & \text{when } |IN| \leq 1 
\end{cases} \quad (7)$$

B. Costate Trajectories and Admissible Controls

The costate equation and its solution are

$$\frac{\partial H}{\partial x} = \dot{\lambda}(t) = \lambda(t) \quad \Rightarrow \quad \lambda(t) = \lambda(0)e^t \quad (8)$$

There are five possible solution trajectories for $\lambda(t)$ that affect the control signal $u(t)$. These different trajectories depend on the initial condition $\lambda(0)$ and its relation to 0 and ±1. The trajectories are shown in Fig. 1, where Case 1 corresponds to $\lambda(0) > 1$, Case 2 corresponds to $0 < \lambda(0) < 1$, Case 3 corresponds to $-1 < \lambda(0) < 0$, Case 4 corresponds to $\lambda(0) < -1$, and Case 5 corresponds to $\lambda(0) = 0$. The time indicated in the figure as $t_1$ is when the trajectories for Case 2 and Case 3 cross ±1.

Since the solution to the costate differential equation is a single growing exponential, $|\lambda(t)|$ never decreases and is always increasing if $\lambda(0) \neq 0$. If $0 < |\lambda(0)| \leq 1$, the initial control will be $u(0) = 0$. As $t$ increases, the costate variable will be

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Fig. 1. The 5 possible costate trajectories that affect the optimal control.

$|\lambda(t)| = 1$ at $t = t_1$. At that time, the control signal will switch to $|u(t_1)| = 1$. It will remain at that value for $t_1 < t \leq T$. If $|\lambda(0)| > 1$, then $|u(t)| = 1$ for $0 \leq t \leq T$. Therefore, if $|u(0)| = 1$, it will never switch to $u(t) = 0$. Control sequences of the form $[+1, 0], [-1, 0], [-1, 0, -1], [-1, 0, +1], etc.$ are excluded from being admissible controls.

The only admissible control sequences for this system are shown in Table 1. The Minimum Principle effectively filters out all other control sequences and provides us with the only admissible ones, regardless of the initial condition or the target state. Other control sequences could obviously be applied to the system, but they would not represent minimum fuel solutions.

### Table 1: Admissible control sequences for minimum fuel control.

<table>
<thead>
<tr>
<th>Case</th>
<th>$u^*(t)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[-1]</td>
<td>No switching</td>
</tr>
<tr>
<td>2</td>
<td>[0, -1]</td>
<td>Switch at $t = t_1$</td>
</tr>
<tr>
<td>3</td>
<td>[0, +1]</td>
<td>Switch at $t = t_1$</td>
</tr>
<tr>
<td>4</td>
<td>[+1]</td>
<td>No switching</td>
</tr>
<tr>
<td>5</td>
<td>[0]</td>
<td>No switching</td>
</tr>
</tbody>
</table>

At the final time, $t = T$, the minimum fuel problem ends. The control that is applied for $t > T$ depends on the performance requirements that are imposed for that time period. If it is desired to maintain the state at the target state, so that $x(t) = x_T$ for $t \geq T$, then for the system model used in this example, the equilibrium control is $u(t) = x_T$ for $t > T$. This control will make $\dot{x}(t) = 0$. The control used to maintain this equilibrium condition is not included in the minimum fuel performance index since $J$ only includes time in the interval $0 \leq t \leq T$.

### C. State Trajectories

The state trajectories can be found by substituting each of the allowed values for the control signal $u(t)$ into the state equation, and then solving the state equation for $x(t)$. This will determine what target states can be reached and whether or
not the reachable target states depend on the initial condition. For the general linear, time-invariant system described by

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(9)

the solution to the differential equation is

\[ x(t_2) = e^{A(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A(t_2-\tau)}Bu(\tau)d\tau \]  

(10)

For the system model used in this example \((A = -1, B = 1)\), and the allowed control values, the state trajectories are given by the following expressions.

- For \(u(t) = 0\)

\[ \dot{x}(t) = -x(t) \quad \Rightarrow \quad x(t_2) = x(t_1)e^{-(t_2-t_1)} \]  

(11)

which indicates that \(x(t)\) decays toward the origin \(x = 0\) in accordance with its natural dynamics given by the open-loop eigenvalue at \(s = -1\).

- For \(u(t) = +1\)

\[ \dot{x}(t) = -(t-1) + 1 \quad \Rightarrow \quad x(t_2) = x(t_1)e^{-(t_2-t_1)} + \int_{t_1}^{t_2} e^{-(t_2-\tau)}d\tau \]  

(12)

\[ x(t_2) = e^{-(t_2-t_1)}[x(t_1) - 1] + 1 \]

showing that the state asymptotically approaches \(x(t) = 1\) at \((t_2 - t_1) \to \infty\), regardless of the initial condition \(x_0\).

- For \(u(t) = -1\)

\[ \dot{x}(t) = -(t-1) - 1 \quad \Rightarrow \quad x(t_2) = x(t_1)e^{-(t_2-t_1)} - \int_{t_1}^{t_2} e^{-(t_2-\tau)}d\tau \]  

(13)

\[ x(t_2) = e^{-(t_2-t_1)}[x(t_1) + 1] - 1 \]

showing that with this input signal the state asymptotically approaches \(x(t) = -1\) at \((t_2 - t_1) \to \infty\).

Figure 2 shows typical state trajectories for the three values of \(u(t)\). From the trajectories that are shown, the following observations can be made:

- If the specified final time \(T\) is finite, then \(u(t) = 0\) cannot be the only control value used if the target state is \(x(T) = x_T = 0\) or if the target state has the opposite sign of the initial state \(x_0\), that is, if \(x_0x_T < 0\).

- The target state has to be in the open interval \(-1 < x_T < 1\) in order for the target state to be reached in a finite time from any initial condition. If the target state were outside that interval, \(|x_T| \geq 1\), there would be some initial conditions that could not reach the target state at all. For example, no state trajectory with initial condition \(x_0 > -1\) can ever reach any target state \(x_T < -1\), and it can only reach \(x_T = -1\) in infinite time. On the other hand, an initial condition \(x_0 > +1\) can reach a target state \(x_0 > x_T > +1\) in a finite time with the optimal control being either \(u = 0\) or \(u = [0, -1]\), depending on the value of \(T\). The control \(u = +1\) would not be used since it has the same fuel cost as \(u = -1\) and yet takes longer to drive the state toward its target value. However, \(|x_T| \geq 1\) does limit valid initial conditions.

- For \(u(t) = +1\) or \(u(t) = -1\), the state trajectory tends toward \(x(t) = u(t)\) as \(t \to \infty\), asymptotically approaching that value. Therefore, the state trajectory cannot cross \(x(t) = +1\) from either direction when \(u(t) = +1\). A similar statement holds for \(x(t) = -1\) with \(u(t) = -1\).

- Any target state \(x_T > -1\) can be reached in a finite time from any initial state \(x_0 > x_T\) with the control \(u(t) = -1\). Therefore, the possible minimum fuel control sequences for this scenario would be \(u(t) = -1\) or \(u(t) = [0, -1]\) depending on the value of \(T\). Likewise, any target state \(x_T < +1\) can be reached in a finite time from any initial state \(x_0 < x_T\) with the control \(u(t) = +1\). Therefore, the possible minimum fuel control sequences in this case would be \(u(t) = +1\) or \(u(t) = [0, +1]\), again depending on the value of \(T\).

\textbf{Example 1:} Let the system model be given by (2), and let the target state and initial condition be \(x(T) = x_T = 0, x(0) = x_0 = 5\). If only \(u(t) = -1\) is used, then the target state can be reached, and it will be reached in minimum time.

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Fig. 2. State trajectories for $u(t) = 0$, $u(t) = +1$, and $u(t) = -1$. 
The performance index is \( J_1 = \int_0^T |-1| \, dt = T \), and the value of \( T \) necessary to reach \( x_T \) from \( x_0 \) with \( u = -1 \) is \( T = \ln [1 + 5] = 1.792 \) seconds. If we specify that \( u(t) = [0, -1] \) and specify the switching time to be \( t_1 = 2 \) seconds, then

\[
J_2 = \int_0^2 |0| \, dt + \int_2^T |-1| \, dt = T - 2
\]

and \( T = 2 + \ln [1 + 5e^{-2}] = 2.517 \) seconds, so \( J_2 = 0.517 \). This is less than the minimum time solution using only \( u(t) = -1 \). If we let the switching time be \( t_1 = 3 \) seconds, then

\[
J_3 = \int_0^3 |0| \, dt + \int_3^T |-1| \, dt = T - 3
\]

and \( T = 3 + \ln [1 + 5e^{-3}] = 3.222 \) seconds, so \( J_3 = 0.222 \). This is less than either of the two previous solutions. This implies that the longer we allow the system to take in order to reach the target state, the lower the cost in terms of fuel to reach the target state. In this example \( J_3 < J_2 < J_1 \). This is generally the case. However there is a scenario in which letting the system take longer to reach the target state actually increases the cost as well as the time. This scenario will be presented later, in Examples 8 and 9.

In each of the cases in the example above, the value of the performance index \( J \) for this system is the length of time that a non-zero control is being used. This is a general result for single-input systems. Therefore,

\[
J = \int_0^T |u(t)| \, dt = \int_0^{t_1} |0| \, dt + \int_{t_1}^T |\pm 1| \, dt
\]

(16)

\[
J = T - t_1
\]

(17)

If \( T = T_{Min} \), the minimum time solution, then the switching time is \( t_1 = 0 \), and \( J = T = T_{Min} \). For \( T \gg T_{Min} \), the performance index approaches a limiting value. If the control law needed to drive \( x_0 \) to \( x_T \) includes \( u(t) = +1 \) or \( u(t) = -1 \), the limiting value of \( J \) depends only on the absolute value of the target state \( x_T \). The limiting value of \( J \) depends neither on the sign of \( x_T \) nor on the value of the control signal used to drive \( x(t) \) toward \( x_T \). In this case the limiting value of the performance index is

\[
J_{Lim} = \max \left[ -\ln(1 + x_T), -\ln(1 - x_T) \right]
\]

(18)

The validity of (18) is seen in the next section in Eqn. (24) and the discussion following. If \( x_T = 0 \), or if \( u(t) = 0 \) will allow the state to decay from \( x_0 \) to \( x_T \) in a finite time, then \( J_{Lim} = 0 \); otherwise, \( J_{Lim} > 0 \).

In order for \( J \) to actually represent a quantity of fuel that is being minimized, the control signal \( u(t) \) must have the units of fuel used per unit time. The integration over time then yields a value for \( J \) having units of total fuel consumed. In these notes, values for \( J \) will be given in each example without any associated units.

**D. Determining the Switching Time \( t_1 \) and the Minimum Time \( T_{Min} \)**

Assume that the target state is \( -1 < x_T < 0 \) and the initial condition is \( x_0 > 0 \) so that the control \( u(t) = 0 \) will not force the system to reach the target state in a finite time. Therefore, either \( u(t) = -1 \) or \( u(t) = [0, -1] \) must be used, depending on the value of \( T \). Assuming that \( T > T_{Min} \) so that the control sequence will be \( u(t) = [0, -1] \), let \( t = t_1 \) be the time at which the control switches from \( u(t) = 0 \) to \( u(t) = -1 \). The initial time will be taken as \( t = 0 \). From (11) and (13) the state trajectory will be

\[
x(t_1) = x_0 e^{-t_1}
\]

(19)

\[
x(T) = x_T = x(t_1) e^{-(T-t_1)} - \int_{t_1}^T e^{-(T-\tau)} \, d\tau = x_0 e^{-T} - e^{-T} \left[ e^{T} - e^{t_1} \right]
\]

(20)

\[
x_T = x_0 e^{-T} - 1 + e^{-(T-t_1)} = x_0 e^{-T} - 1 + e^{-T} e^{t_1}
\]

(21)

\[
e^{t_1} = e^{T} (x_T + 1) - x_0 \quad \text{or} \quad e^{-(T-t_1)} = x_T + 1 - x_0 e^{-T}
\]

(22)

The first expression in (22) can be solved for the relationship between \( t_1, x_0, \) and \( x_T \).

\[
t_1 = \ln \left[ e^{T} (x_T + 1) - x_0 \right]
\]

(23)

An alternate equation for \( t_1 \) can be developed by solving the second expression in (22) for \( T - t_1 \).

\[
T - t_1 = J = -\ln \left[ x_T + 1 - x_0 e^{-T} \right]
\]

(24)
If \( T \gg 1 \), then the exponential on the right side of (24) will be negligible, and \( T - t_1 \approx -Ln [x_T + 1] \). If the target state is the origin, then \( T - t_1 \approx 0 \) for large \( T \). This means that \( u(t) = 0 \) is used for most of the time, so the system coasts for as long as possible. Since \( J = T - t_1 \), for the control sequence \( u(t) = [0, -1] \), the cost can be made arbitrarily small (for \( x_T = 0 \)) by letting the final time be arbitrarily large. However, there is no solution to the optimal control problem if the target state is the origin and \( T \) is free. If \( x_T \neq 0 \), then the limiting value of the performance index will be positive, with its value depending only on \( x_T \).

If the final time is fixed, then (24) can be solved for the switching time \( t_1 \)

\[
t_1 = T + Ln [x_T + 1 - x_0 e^{-T}] \tag{25}
\]

The final time has to be fixed at a value greater than or equal to the minimum time \( T_{Min} \) in order for there to be any solution to the minimum fuel control problem, and \( T > T_{Min} \) is necessary for \( t_1 > 0 \). The time optimal solution can be obtained from (19) and (20) by setting \( t_1 = 0 \), and it is given by

\[
x_T = x_0 e^{-T} - e^{-T} [e^T - 1] = e^{-T} [x_0 + 1] - 1 \tag{26}
\]

\[
e^T [x_T + 1] = [x_0 + 1] \tag{27}
\]

\[
T^* = T_{Min} = Ln \left[ \frac{x_0 + 1}{x_T + 1} \right] \tag{28}
\]

The expressions developed above for the performance index and minimum time are also valid for \( 0 < x_T < x_0 \) and when \(-1 < x_T < x_0 \leq 0 \). The governing rule for this condition is that \( (x_0 + 1)(x_T + 1) > 0 \) and \(|(x_0 + 1)| > |(x_T + 1)| \) and \( x_T > -1 \). When the third condition, based on requiring the target state to be in the open interval \(-1 < x_T < 1 \), is included, this simplifies to \( x_0 > x_T \). Likewise, the expressions for the switching time are valid as long as the relationship between \( x_0 \), \( x_T \), and \( T \) are such that switching occurs. Otherwise, \( t_1 = 0 \). The expressions are summarized in Table 2.

### Table 2: Summary of equations for \( u = -1 \) or \( u = [0, -1] \).

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Equation</th>
<th>Eqn. No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Switching Time</td>
<td>( t_1 = Ln [e^T (x_T + 1) - x_0] )</td>
<td>23</td>
</tr>
<tr>
<td>Performance Index</td>
<td>( T - t_1 = J = -Ln [x_T + 1 - x_0 e^{-T}] )</td>
<td>24</td>
</tr>
<tr>
<td>Switching Time (alternate)</td>
<td>( t_1 = T + Ln [x_T + 1 - x_0 e^{-T}] )</td>
<td>25</td>
</tr>
<tr>
<td>Minimum Time</td>
<td>( T^* = T_{Min} = Ln \left[ \frac{x_0 + 1}{x_T + 1} \right] )</td>
<td>28</td>
</tr>
</tbody>
</table>

Similar expressions can be developed for the case where \( x_0 < 0 \) and \( 0 \leq x_T < 1 \), which requires \( u = +1 \). The general rule for this case is \((x_0 - 1)(x_T - 1) > 0 \) and \(|(x_0 - 1)| > |(x_T - 1)| \) and \( x_T < 1 \), which simplifies to \( x_0 < x_T \) when the restriction on the target state is imposed. The expressions are developed in the same way and are given in Table 3.

### Table 3: Summary of equations for \( u = +1 \) or \( u = [0, +1] \).

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Switching Time</td>
<td>( t_1 = Ln [-e^T (x_T - 1) + x_0] )</td>
</tr>
<tr>
<td>Performance Index</td>
<td>( T - t_1 = J = -Ln [-x_T + 1 + x_0 e^{-T}] )</td>
</tr>
<tr>
<td>Switching Time (alternate)</td>
<td>( t_1 = T + Ln [-x_T + 1 + x_0 e^{-T}] )</td>
</tr>
<tr>
<td>Minimum Time</td>
<td>( T^* = T_{Min} = Ln \left[ \frac{x_0 - 1}{x_T - 1} \right] )</td>
</tr>
</tbody>
</table>

**Example 2**: Using the same initial condition \( x_0 = 5 \) and target state \( x_T = 0 \) as in Example 1, Table 4 shows the switching time and total cost for several different values of \( T \), including the minimum time \( T = 1.792 \).

### Table 4: Switching time and total cost as a function of final time \( T \).

<table>
<thead>
<tr>
<th>( T )</th>
<th>( t_1 )</th>
<th>( J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.792</td>
<td>0</td>
<td>1.792</td>
</tr>
<tr>
<td>2</td>
<td>0.871</td>
<td>1.129</td>
</tr>
<tr>
<td>2.5</td>
<td>1.972</td>
<td>0.528</td>
</tr>
<tr>
<td>3</td>
<td>2.714</td>
<td>2.86 \times 10^{-1}</td>
</tr>
<tr>
<td>5</td>
<td>4.97</td>
<td>3.43 \times 10^{-2}</td>
</tr>
<tr>
<td>10</td>
<td>9.9998</td>
<td>2.2703 \times 10^{-4}</td>
</tr>
</tbody>
</table>
The entries in the table clearly show the reduction in $J$ as $T$ increases. For large $T$, $t_1 \approx T$ and $J \approx 0$. Figure 3 graphically shows the relationship between the specified final time $T$ and the performance index $J = T - t_1$. The vertical dashed line in the figure is at $T = T_{\text{Min}} = 1.792$ seconds. The graph clearly shows that when $x_T = 0$, the limiting value of the performance index is $J_{\text{Lim}} = 0$, and that $J$ decreases fairly rapidly with increasing $T$ above $T_{\text{Min}}$. For a specific value of $T$, the corresponding value of $J$ can be obtained from the graph, so the cost of operating the system from $x_0$ can be determined in advance. The value of $J = 0.749$ for $T = 2.25$ seconds is shown for illustration.

**E. The Switching Curve and Closed-Loop Control**

Once the target state $x_T$, the initial condition $x_0$, and the final time $T$ are known, the value of the switching time $t_1$ can be computed from the appropriate entry in Table 2 or Table 3. The system can then be operated with $u(t) = 0$ for $0 \leq t \leq t_1$ and $u(t) = -1$ (or $+1$) for $t_1 < t \leq T$. The only requirement is to keep track of time. This is open-loop control since the current value of the state $x(t)$ is not used to determine the control signal. It would be preferable to use closed-loop control for the usual reasons of reducing the effects of uncertainty and disturbances. This can be done by comparing the current state with the value of a switching curve.

Assume that the initial condition and target state satisfy the condition $x_0 > x_T$, requiring the non-zero portion of the minimum fuel control to be $u(t) = -1$. Then the switching curve is the state trajectory that reaches the target state $x(T) = x_T$ at the final time $T$ using only the control $u(t) = -1$. Call this switching curve $z(t)$. The expression for $z(t)$ has the same form as (13), and is given by

$$z(t) = e^{-t} (z_0 + 1) - 1$$

(29)

where the initial condition is

$$z_0 = e^T (x_T + 1) - 1$$

(30)
Substituting (30) into (29) and letting \( t = T \) yields the result that \( z(T) = x_T \). Note that the switching curve depends only on the target state \( x_T \) and the final time \( T \), not on the initial condition \( x_0 \). The corresponding equations for the condition \( x_0 < x_T \), with \( u(t) = +1 \), are

\[
\begin{align*}
   z(t) &= e^{-t}(z_0 - 1) + 1 \quad (31) \\
   z_0 &= e^T(x_T - 1) + 1 \quad (32)
\end{align*}
\]

The closed-loop control law is shown in Table 5, which can be simplified as shown in (33) and (34).

<table>
<thead>
<tr>
<th>( x_0 &gt; x_T )</th>
<th>( x_0 &lt; x_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(t) = 0 ), if ( x(t) &lt; z(t) )</td>
<td>( u(t) = 0 ), if ( x(t) &gt; z(t) )</td>
</tr>
<tr>
<td>( u(t) = -1 ), if ( x(t) \geq z(t) )</td>
<td>( u(t) = +1 ), if ( x(t) \leq z(t) )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
   u(t) &= 0, \quad \text{if } (x_0 - x_T) [x(t) - z(t)] < 0 \quad (33) \\
   u(t) &= -SGN(x_0 - x_T), \quad \text{if } (x_0 - x_T) [x(t) - z(t)] \geq 0 \quad (34)
\end{align*}
\]

The minimum fuel control law allows this system—which is open-loop stable—to respond naturally for as long as possible with zero control. A non-zero control is applied only at the time necessary to force the state so that it arrives at the target state at the specified final time. Figure 4 shows the switching curve for \( T = 2.25 \) seconds and \( x_T = 0 \). The initial condition for the state is \( x_0 = 5 \). The heavy line in the figure is the actual trajectory that the state variable would follow, reaching the origin at the specified value of \( T \). The switching takes place at \( t_1 = 1.501 \) seconds.
F. Illustrative Examples

1) Overview of the Examples: Several examples will be presented in this section to illustrate the use of the various expressions developed in the previous sections and to point out one scenario that has a peculiar twist to it. These examples are certainly not exhaustive, but they do show the major aspects of minimum fuel control for this first-order stable system. Changing the signs of the initial condition \( x_0 \) and the target state \( x_T \) only changes the sign of the control signal \( u(t) \); there is no change in the fundamental properties of the control or state trajectories. The examples to be presented are listed below and summarized in Table 6.

Ex. 3. \( x_0 > 0, x_T \in (-1, 0), T > T_{Min} \)
Ex. 4. \( x_0 > 0, x_T \in (-1, 0), T = T_{Min} \)
Ex. 5. \( x_0 < 0, x_T \in (0, +1), T > T_{Min} \)
Ex. 6. \( x_0 > 0, x_T \in (0, x_0), T_{Min} < T < T_{u-0} \)
Ex. 7. \( x_0 > 0, x_T \in (0, x_0), T_{Min} < T < u_{-0} \leq T \)
Ex. 8. \( x_0 > 0, x_T \in (x_0, +1), T > T_{Min}, T \) fixed in value
Ex. 9. \( x_0 > 0, x_T \in (x_0, +1), T > T_{Min}, T \) set equal to \( T_{Min} \)

The parameter \( T_{u-0} \) is the time required for the state \( x(t) \) to decay from \( x_0 \) to \( x_T \) only using the control \( u(t) = 0 \). This can only occur for the situation defined by \( x_T \neq 0 \) and \( x_0x_T > 0 \) and \( |x_0| > |x_T| \). If \( T_{u-0} \leq T \), then \( u(t) = 0 \) is the minimum fuel control, and \( T \) will be reduced to \( T_{u-0} \) if the specified \( T \) is greater than \( T_{u-0} \).

Examples 8 and 9 illustrate the scenario that has the peculiar twist that was previously mentioned. In Example 2, shown on page 6, it was seen that increasing the value of the final time \( T \) reduces the amount of fuel used to drive the state from \( x_0 \) to \( x_T \). This characteristic will also be seen in Examples 3–7 in this section. For the conditions in Examples 8 and 9, however, increasing the final time actually increases the amount of fuel used. The true minimum fuel solution for these conditions is also the minimum time solution. Those two examples will illustrate the results when \( T \) is required to be fixed at its specified value and when it can be decreased to equal the optimal minimum time value.

<table>
<thead>
<tr>
<th>Example</th>
<th>( x_0 )</th>
<th>( x_T )</th>
<th>( T ) (sec)</th>
<th>( t_1 ) (sec)</th>
<th>( J )</th>
<th>( T_{Min} ) (sec)</th>
<th>( J_{Lim} )</th>
<th>( u^*(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>-0.5</td>
<td>2.5</td>
<td>1.129</td>
<td>1.371</td>
<td>2.079</td>
<td>0.693</td>
<td>[0, -1]</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>-0.5</td>
<td>2.079</td>
<td>0</td>
<td>2.079</td>
<td>2.079</td>
<td>0.693</td>
<td>[-1, 1]</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.5</td>
<td>2.5</td>
<td>1.129</td>
<td>1.371</td>
<td>2.079</td>
<td>0.693</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>6</td>
<td>0.75</td>
<td>0.25</td>
<td>0.5</td>
<td>0.271</td>
<td>0.229</td>
<td>0.336</td>
<td>0</td>
<td>[0, -1]</td>
</tr>
<tr>
<td>7</td>
<td>0.75</td>
<td>0.25</td>
<td>1.099</td>
<td>N/A</td>
<td>0</td>
<td>0.336</td>
<td>0</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>8</td>
<td>0.35</td>
<td>0.75</td>
<td>1.75</td>
<td>0.581</td>
<td>1.169</td>
<td>0.956</td>
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<tr>
<td>9</td>
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<td>0</td>
<td>0.956</td>
<td>0.956</td>
<td>1.386</td>
<td>[1]</td>
</tr>
</tbody>
</table>

2) Description of the Examples: The examples will be described on the following pages. The important points of each example will be highlighted to illustrate how that example is similar to or different from the other examples. For each of the examples, three plots are provided. The upper left graph shows the state trajectory \( x(t) \) (blue line) and switching curve \( z(t) \) (red line). The upper right graph shows the control signal \( u(t) \). In each case, it is assumed that the goal is to maintain the state at the target value for \( t \geq T \). Therefore, the control signal takes on its equilibrium value \( u(t) = x_T \) for \( t > T \). The bottom graph shows the relationship between the value of the performance index \( J = T - t_1 \) and the final time \( T \). In each of the examples, the relationships between the initial state, the final state, and the final time should be noted, as well as the relationship between the final time and the minimum time \( T_{Min} \), and the relationship between the state \( x(t) \) and the switching curve \( z(t) \).
**Example 3:** $x_0 = 3$, $x_T = -0.5$, $T = 2.5$ sec.

The graphs for this example are shown in Fig. 5. Since $x_0 > 0$ and $x_T < 0$, the control signal must have the value $u(t) = -1$ for at least part of the time since $u(t) = 0$ cannot force $x(t)$ to change sign. From (28), $T_{Min} = \ln \left( \frac{x_0 + 1}{x_T + 1} \right) = 2.079$ sec., so it is clear that $T > T_{Min}$. This allows the control signal to be $u(t) = 0$ for part of the time. From (23) the time at which the control switches from $u(t) = 0$ to $u(t) = -1$ is $t_1 = \ln \left( e^{T} (x_T + 1) - x_0 \right) = 1.129$ sec. The state response is seen to decay naturally from $t = 0$ to $t = t_1$ with the applied control being $u(t) = 0$. At time $t = t_1$, the state trajectory intersects the switching curve trajectory, and the control switches from $u(t) = 0$ to $u(t) = -1$. The switching curve and state reach the desired value $x_T = -0.5$ at $t = T = 2.5$ sec. It is assumed that the goal is for the state to be held at that final value for $t > T$, so the control signal is switched to $u(t) = 0$ at $t = T$. The value of the performance index for this example is $J = T - t_1 = 1.371$. The bottom plot in Fig. 5 shows that relationship between the final time $T$ and the value of $J$. In that case, the switching time would be $t_1 = 0$ (the control starts at $u(t) = -1$), and the value of the performance index would be $J = T - t_1 = 2.079 = T_{Min}$. The plot shows that increasing $T$ decreases the value of $J$, but the performance index does reach a limiting value. Since $|x_T| < 1$, the control signal can take on this equilibrium value under the specified control constraints. The control signal is seen to be piecewise constant, switching values once at $t = t_1$ and again at $t = T$ in order to establish equilibrium at $x = x_T$.

The value of the performance index for this example is $J = T - t_1 = 1.371$. The bottom plot in Fig. 5 shows that relationship between the final time $T$ and the value of $J$. The maximum cost would be incurred if $T = T_{Min} = 2.079$ sec. In that case, the switching time would be $t_1 = 0$ (the control starts at $u(t) = -1$), and the value of the performance index would be $J = T - t_1 = 1.371 = T_{Min}$. The plot shows that increasing $T$ decreases the value of $J$, but the performance index does reach a limiting value. Since $|x_T| > 0$ and $u(t) \neq 0$ is required, the limiting value of $J$ is non-zero. In accordance with (18), $J_{Lim} = -\ln (1 + x_T) = 0.693$.  

![Graphs showing the state, control signal, and performance index for Example 3.](image-url)
Example 4: \(x_0 = 3, x_T = -0.5, T = 2.079\) sec.

The graphs for this example are shown in Fig. 6. As in Example 3, the control signal must have the value \(u(t) = -1\) for at least part of the time since \(x_0 > 0\) and \(x_T < 0\) so that \(u(t) = 0\) cannot drive the system state to the target state. The minimum time depends only on the initial and target states, so it has the same value as before: \(T_{\text{Min}} = 2.079\) sec. The final time in this example is chosen to be equal to the minimum time, so the control signal cannot be \(u(t) = 0\) any time during the trajectory. Thus, the time at which the control switches from \(u(t) = 0\) to \(u(t) = -1\) is \(t_1 = 0\) sec. (the control starts at \(u(t) = -1\)). There is no natural decaying of the state in this situation. The switching curve has the initial condition \(z_0 = x_0\), so it and the state reach the desired value \(x_T = -0.5\) at \(t = T = 2.079\) sec. As before, it is assumed that the goal is for the state to be held at that final value for \(t > T\), so the control signal is switched to \(u(t) = x_T = -0.5\) at \(t = T\). This makes \(\dot{x}(t) = 0\) for \(t \geq T\).

The value of the performance index for this example is \(J = T - t_1 = 2.079\), which is obviously higher than in Example 3. Since \(T\) is the independent variable in the bottom plot of the figure—cost vs. final time—and the cost depends on \(T\), \(x_0\), and \(x_T\), the curve showing that relationship for this example is the same as in the previous example since \(x_0\) and \(x_T\) are unchanged. As seen in the bottom plot, the maximum cost has been incurred since \(T = T_{\text{Min}} = 2.079\) sec. Since the limiting value of \(J\) depends only on \(|x_T|\) (except when \(u(t) = 0\) is a valid control by itself), that limiting value is the same as in Example 3, namely, \(J_{\text{Lim}} = -\ln (1 + x_T) = 0.693\).
Example 5: $x_0 = -3, x_T = 0.5, T = 2.5 \text{ sec.}$

The graphs for this example are shown in Fig. 7. This example is just the mirror image of Example 3. The signs of the initial condition and target state are both reversed from the previous example. The values of switching time, minimum time, performance index, and limiting value for the performance index are unchanged, although the times are computed from different expressions because of the changes in signs of $x_0$ and $x_T$. The correct expressions for $t_1$, $T_{Min}$, and $J$ are obtained from Table 3, and the expression for $J_{Lim}$ is $J_{Lim} = -Ln (1 - x_T)$. The control sequence is $u(t) = [0, +1]$ rather than $u(t) = [0, -1]$ as it was in Example 3. Reversing the signs on both $x_0$ and $x_T$ has no effect except to change the sign on the non-zero portion of $u(t)$.

For any of these examples, the bottom plot can be used to determine the value of the final time $T$ needed in order to achieve a particular value of the performance index $J = J_0$ in the interval $T_{Min} \geq J > J_{Lim}$. This can be done by substituting the desired value $J_0$ into (24), or the corresponding expression in Table 3, and solving for the value of $T$. The resulting value of $T$ is

$$T = -Ln \left[ \frac{1 + x_T - e^{-J_0}}{x_0} \right], \quad x_0 \neq 0$$

(35)

If $x_0 = 0$, then $J = J_{Lim} = \max \left[ -Ln (1 + x_T), -Ln (1 - x_T) \right]$ regardless of the value of $T$.

Fig. 7. Results from Example 5: $x_0 = -3, x_T = 0.5, T = 2.5 \text{ sec.}$
Example 6: $x_0 = 0.75$, $x_T = 0.25$, $T = 0.5$ sec.

The graphs for this example are shown in Fig. 8. This example and the following one illustrate the situation when the target state can be reached from the initial state using $u(t) = 0$ only if $T$ is sufficiently long. Since the open-loop system is stable, $x_0$ will decay naturally to $x_T$ in a finite time if $|x_0| > |x_T| > 0$. The time needed for this decay is $T_{u-0} = \ln(x_0/x_T)$. If the specified final time $T$ is greater than or equal to $T_{u-0}$, then the minimum fuel control would be $u(t) = 0$, and the cost would be $J = 0$. If $T < T_{u-0}$, then $u(t) = +1$ or $u(t) = -1$ is required, depending on the sign of $x_0$. For the values given in this example, $T_{u-0} = \ln(3) = 1.099$ sec., so with $T = 0.5$ sec. the control signal cannot be $u(t) = 0$ by itself. The switching time is computed in the normal fashion, and the control signal switches from $u(t) = 0$ to $u(t) = -1$ at $t = t_1$. The top two graphs in Fig. 8 show the state trajectory, switching curve, and control signal.

As with the previous examples, the cost decreases with increasing $T$, as seen in the bottom graph, with a limiting value of $J = 0$. Unlike the examples shown in Fig. 3 and Figs. 5–7, however, the limiting value is not reached asymptotically in this example. The graph of $J$ vs. $T$ in Fig. 8 appears to be approaching a negative value as $T$ increases. The limiting value $J_{Lim} = 0$ is reached at $T = T_{u-0}$ instead of as $T \to \infty$.

$$J = -\ln\left(\frac{x_T}{x_0} + 1\right) = -\ln\left(x_T + 1 - x_0 e^{-\ln(x_0/x_T)}\right)$$

$$J = -\ln\left(x_T + 1 - x_0 e^{\ln(x_T/x_0)}\right) = -\ln\left(x_T + 1 - x_0 \cdot \frac{x_T}{x_0}\right)$$

For $T > T_{u-0}$, the switching time $t_1$ is greater than the final time. This would make the performance index negative, which is not a valid situation. Therefore, for $T \geq T_{u-0}$, $T \to T_{u-0}$, no switching occurs, $u(t) = 0$, and $J = 0$. This is illustrated in the next example.

![Graphs showing state trajectory, switching curve, and control signal for Example 6.]

Fig. 8. Results from Example 6: $x_0 = 0.75$, $x_T = 0.25$, $T = 0.5$ sec.
Example 7: \(x_0 = 0.75, \ x_T = 0.25, \ T = 1.099\) sec.

The graphs for this example are shown in Fig. 9. The initial state and target state are the same as in Example 6, but now the final time has been increased to \(T = T_{u-0} = 1.099\) sec. The system state can decay naturally from the initial state to the final state, so no control is necessary. The optimal minimum fuel control is \(u(t) = 0\), and the corresponding cost is \(J = 0\). The state and control trajectories are shown in the top two graphs. There is no switching curve shown in the figure since \(z(t)\) would correspond to the control signal \(u(t) = -1\), and the only control used in this scenario is \(u(t) = 0\). The state \(x(t)\) will not coincide with \(z(t)\) anywhere except at \(t = T\), where they both have the value \(x(T) = z(T) = x_T\).

If the final time is specified at a larger value, \(T > T_{u-0}\), then the state will still decay from \(x_0\) to \(x_T\) in \(T_{u-0}\) seconds. There is no way to increase that length of time and still implement a minimum fuel control law. Since \(u(t) = [+1, 0]\) and \(u(t) = [+1, 0, -1]\) are not valid control sequences for this system with the minimum fuel performance index, there is no way to move the state away from the target state in order for the trajectory to take a longer period of time. When the initial and target states are such that \(|x_0| > |x_T| > 0\), then \(T_{u-0}\) is the maximum time that is acceptable for minimum fuel control. Therefore, \(T_{Min} \leq T \leq T_{u-0}\) is required in this situation.

![Graphs showing state and control trajectories for Example 7.](image-url)

**Fig. 9.** Results from Example 7: \(x_0 = 0.75, \ x_T = 0.25, \ T = T_{u-0} = 1.099\) sec.
Example 8: $x_0 = 0.35$, $x_T = 0.75$, $T = 1.75$ sec.

The graphs for this example are shown in Fig. 10. Examples 8 and 9 illustrate the scenario mentioned previously that has a peculiar twist to it, relative to the other scenarios studied. This scenario is characterized by the fact that the control signal $u(t) = 0$ allows the system state $x(t)$ to move in the direction away from the target state. The target state can still be reached using a control sequence of $u(t) = [0, +1]$ or $u(t) = [0, -1]$, depending on the sign of $x_0$. However, the true minimum fuel control for this scenario is also the minimum time control, $u(t) = +1$ or $u(t) = -1$. It actually takes less fuel to force the system to move from $x_0$ to $x_T$ as quickly as possible rather than allowing the system to coast. The reason for this is that when the system is coasting with $u(t) = 0$, it is coasting in the wrong direction. When the control signal $|u(t)| = 1$ is finally applied, the state is farther away from $x_T$ than it initially was, and the non-zero control has to be applied for a longer period of time. Thus, the cost goes up with increasing $T$.

The relationship between $x_0$ and $x_T$ to have this scenario is one of the following:

\begin{align}
1 & > x_T > x_0 > 0, \quad x_0 > 0 \quad (39) \\
-1 & < x_T < x_0 < 0, \quad x_0 < 0 \quad (40)
\end{align}

In this example, the final time is set at $T = 1.75$ sec., which is longer than the minimum time $T_{\text{Min}} = 0.956$ sec. However, this example forces the specified final time to be used for the simulation. Therefore, the initial control is $u(t) = 0$. The switching time $t_1$ is computed in the normal fashion, and the control switches to $u(t) = +1$ at $t = t_1$. The state will reach the target state at the specified final time, as seen in the upper left graph of Fig. 10.

The bottom graph in the figure shows the increase in cost with increasing $T$. This is opposite of what was seen in all the other examples, and this characteristic of the $J$ vs. $T$ curve is specific to this scenario. For the specified final time of $T = 1.75$ sec., the cost is $J = 1.169$. \phantomsection\ref{fig:example8}
**Example 9:** $x_0 = 0.35$, $x_T = 0.75$, $T = 0.956$ sec.

The graphs for this example are shown in Fig. 11. The same values for initial and target states are used here as were used in Example 8. In this example, the final time $T$ is allowed to be the minimum time value of $T_{Min} = 0.956$ sec. The only control that is used is $u(t) = +1$. The switching time is $t_1 = 0$ sec., since this is equivalent to a minimum time problem, so the cost is $J = T - t_1 = T_{Min} = 0.956$. This cost is lower than the value achieved in the previous example, and the target state was reached more rapidly as well. This illustrates the paradox of this particular scenario—a faster response time requires less fuel.

With this particular scenario, $1 > |x_T| > |x_0| > 0$, the benefits of the shortest possible response time and the least amount of fuel consumed are both achieved with the same control law, $u(t) = SGN [x_0]$. The bottom graph in the figure illustrates that the minimum time solution is also the minimum fuel solution for this set of conditions.

![Graphs for Example 9](image-url)