Lecture 7

Tree and Array Multipliers

Required Reading

Behrooz Parhami,
Computer Arithmetic: Algorithms and Hardware Design

Chapter 11, Tree and Array Multipliers
Chapter 12.5, The special case of squaring

Note errata at:
http://www.ece.ucsb.edu/~parhami/text_comp_arit_1ed.htm#errors

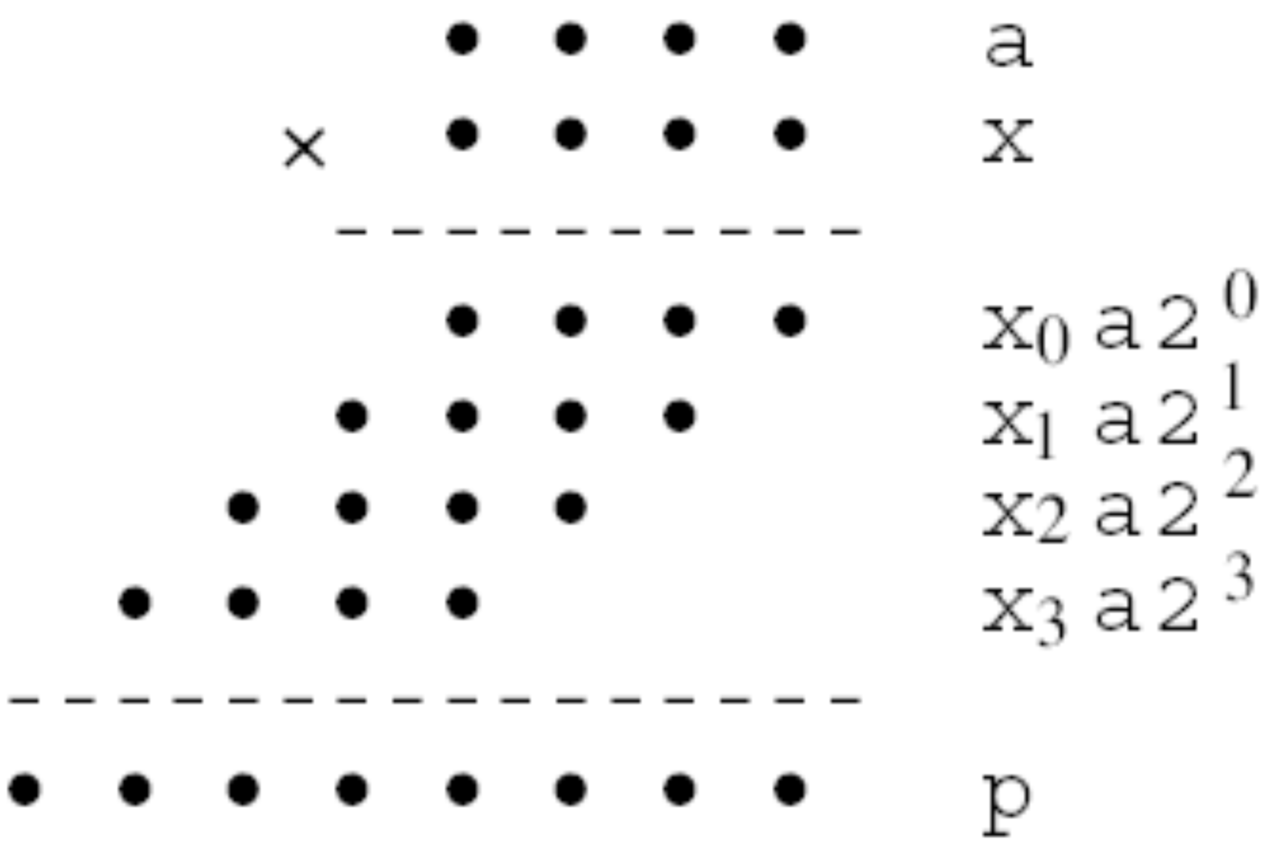
“At least one good reason
for studying multiplication and division is that
there is an infinite number of ways
of performing these operations
and hence there is an infinite number of PhDs
(or expenses-paid visits to conferences in the USA)
to be won from inventing new forms of multiplier.”

Alan Clements
The Principles of Computer Hardware, 1986

Notation

a	Multiplicand	$a_{k-1}a_{k-2} \dots a_1\ a_0$
x	Multiplier	$x_{k-1}x_{k-2} \dots x_1\ x_0$
p	Product $(a \cdot x)$	$p_{2k-1}p_{2k-2} \dots p_2\ p_1\ p_0$

Multiplication of two 4-bit unsigned binary numbers in dot notation



Basic Multiplication Equations

$$p = a \cdot x \qquad\qquad x = \sum_{i=0}^{k-1} x_i \cdot 2^i$$

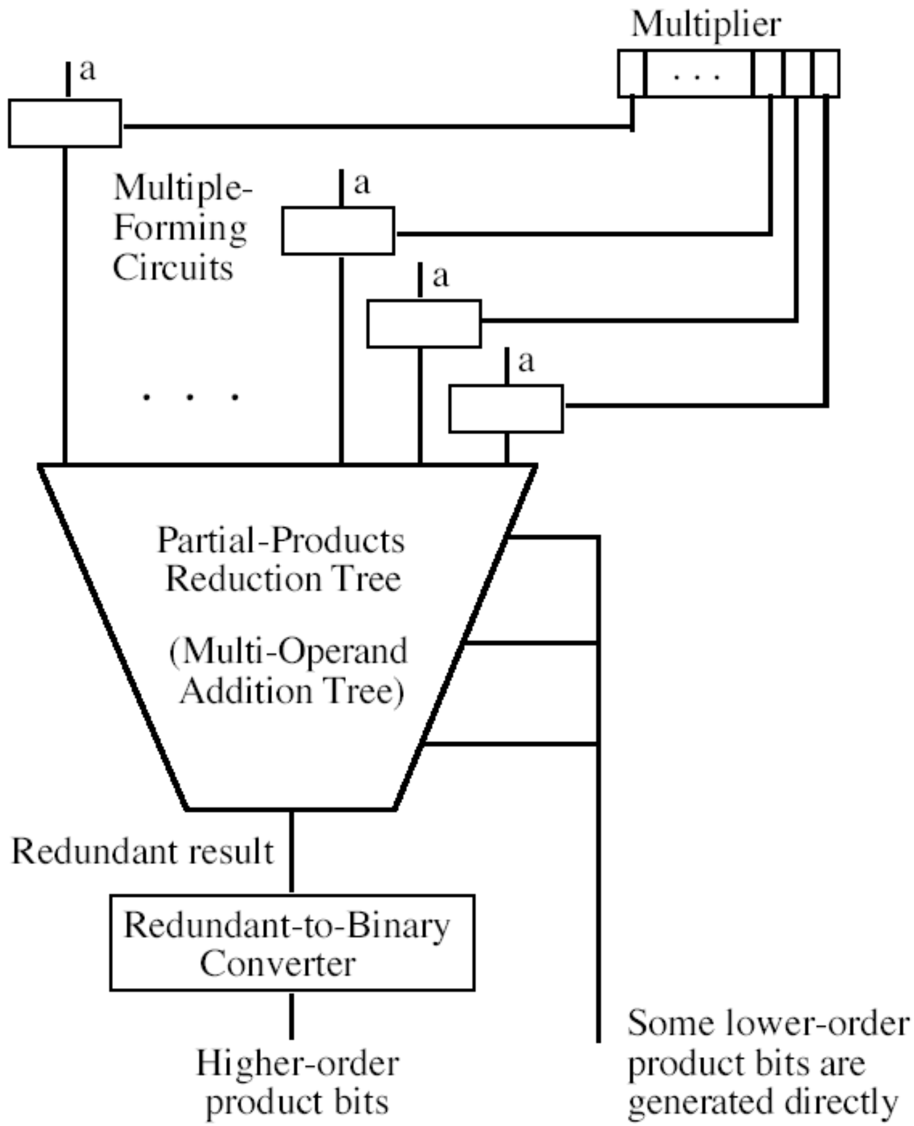
$$p = a \cdot x = \sum_{i=0}^{k-1} a \cdot x_i \cdot 2^i =$$

$$= x_0 a 2^0 + x_1 a 2^1 + x_2 a 2^2 + \ldots + x_{k-1} a 2^{k-1}$$

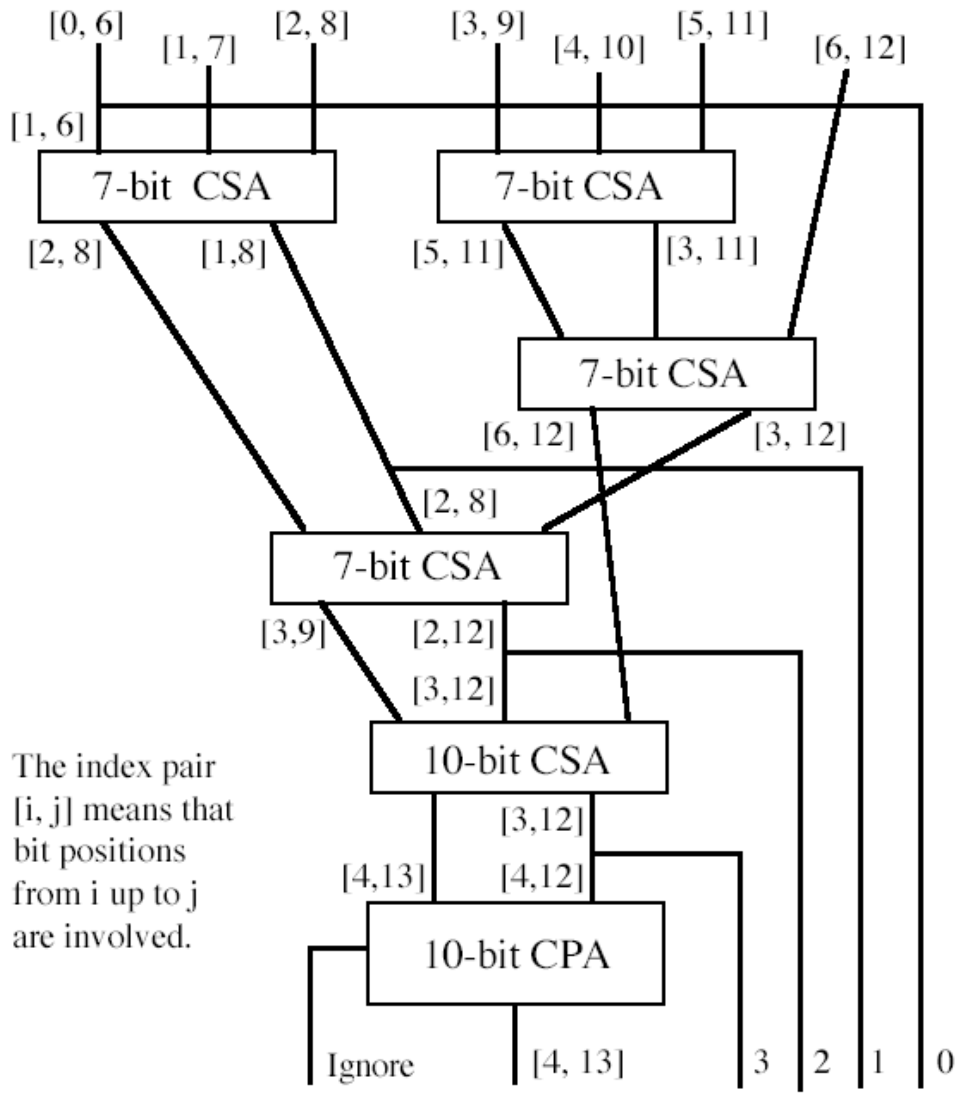
Unsigned Multiplication

						a_4	a_3	a_2	a_1	a_0
					x	x_4	x_3	x_2	x_1	x_0
$ax_0\ 2^0$						a_4x_0	a_3x_0	a_2x_0	a_1x_0	a_0x_0
$ax_1\ 2^1$		+			a_4x_1	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
$ax_2\ 2^2$				a_4x_2	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
$ax_3\ 2^3$			a_4x_3	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
$ax_4\ 2^4$		a_4x_4	a_3x_4	a_2x_4	a_1x_4	a_0x_4				
	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0

Full tree multiplier - general structure

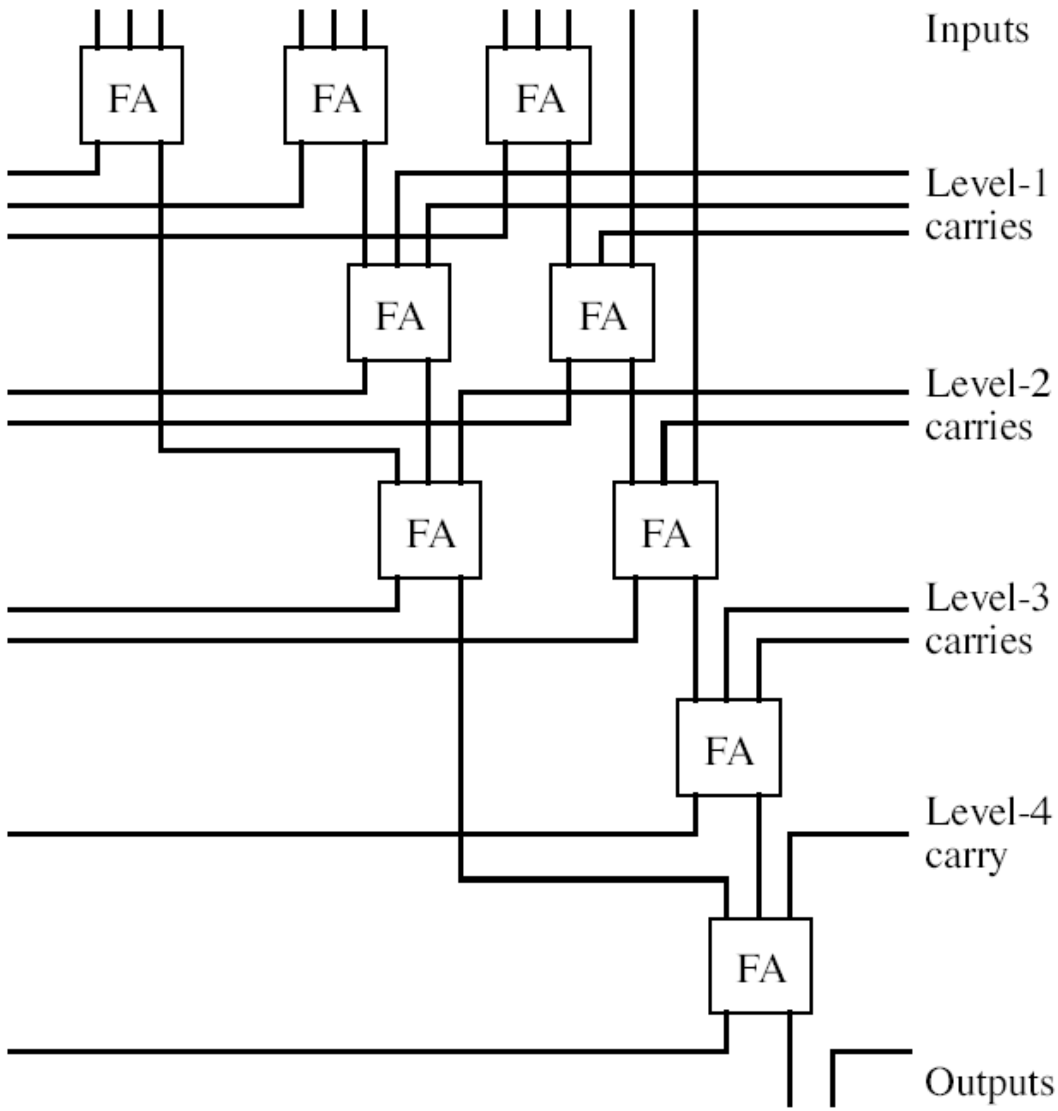


7 x 7 tree multiplier

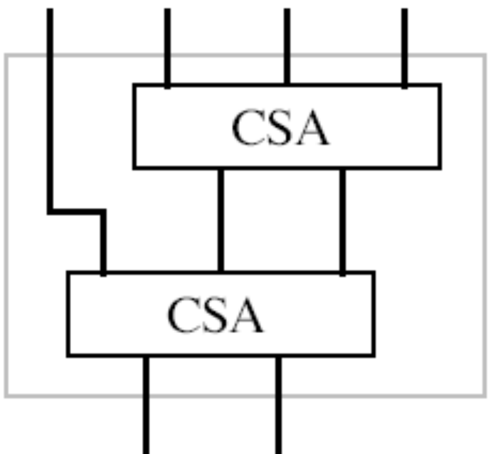


The index pair [i, j] means that bit positions from i up to j are involved.

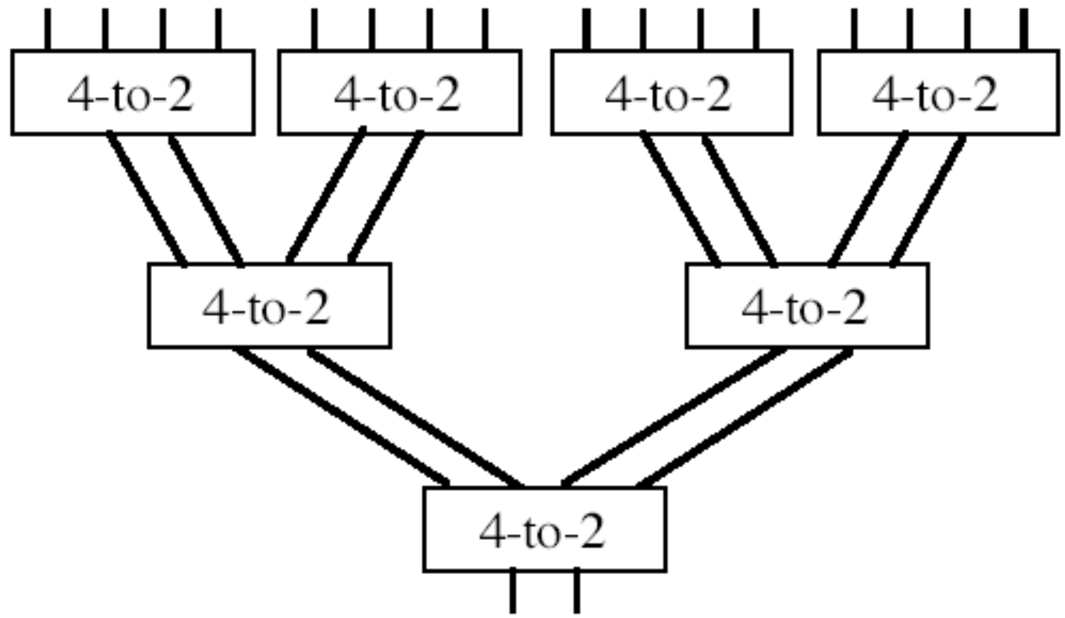
A slice of a balanced-delay tree for 11 inputs



Tree multiplier
with a more regular structure



4-to-2 reduction module
implemented with two
levels of (3; 2)-counters



Unsigned vs. Signed Multiplication

Unsigned		Signed	
1111	15	1111	-1
x 1111	x 15	x 1111	x -1
11100001	225	00000001	1

2's Complement Multiplication (1)

	$\mathbf{-2^4}$	2^3	2^2	2^1	2^0
	$\mathbf{a_4}$	a_3	a_2	a_1	a_0
x	$\mathbf{x_4}$	x_3	x_2	x_1	x_0

$\Updownarrow$ ≡

	$\mathbf{2^4}$	2^3	2^2	2^1	2^0
	$\mathbf{-a_4}$	a_3	a_2	a_1	a_0
x	$\mathbf{-x_4}$	x_3	x_2	x_1	x_0

2's Complement Multiplication (2)

						$-a_4$	a_3	a_2	a_1	a_0
					x	$-x_4$	x_3	x_2	x_1	x_0
						$-a_4x_0$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$-a_4x_1$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$-a_4x_2$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$-a_4x_3$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$-a_3x_4$	$-a_2x_4$	$-a_1x_4$	$-a_0x_4$				
	$-p_9$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2's Complement Multiplication (3)

$-\mathbf{p_9}$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
$\mathbf{2^9}$	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

$\equiv$

$\mathbf{p_9}$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
$\mathbf{-2^9}$	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2's Complement Multiplication (4)

$$\overline{z} = 1 - z$$

$$z = 1 - \overline{z}$$

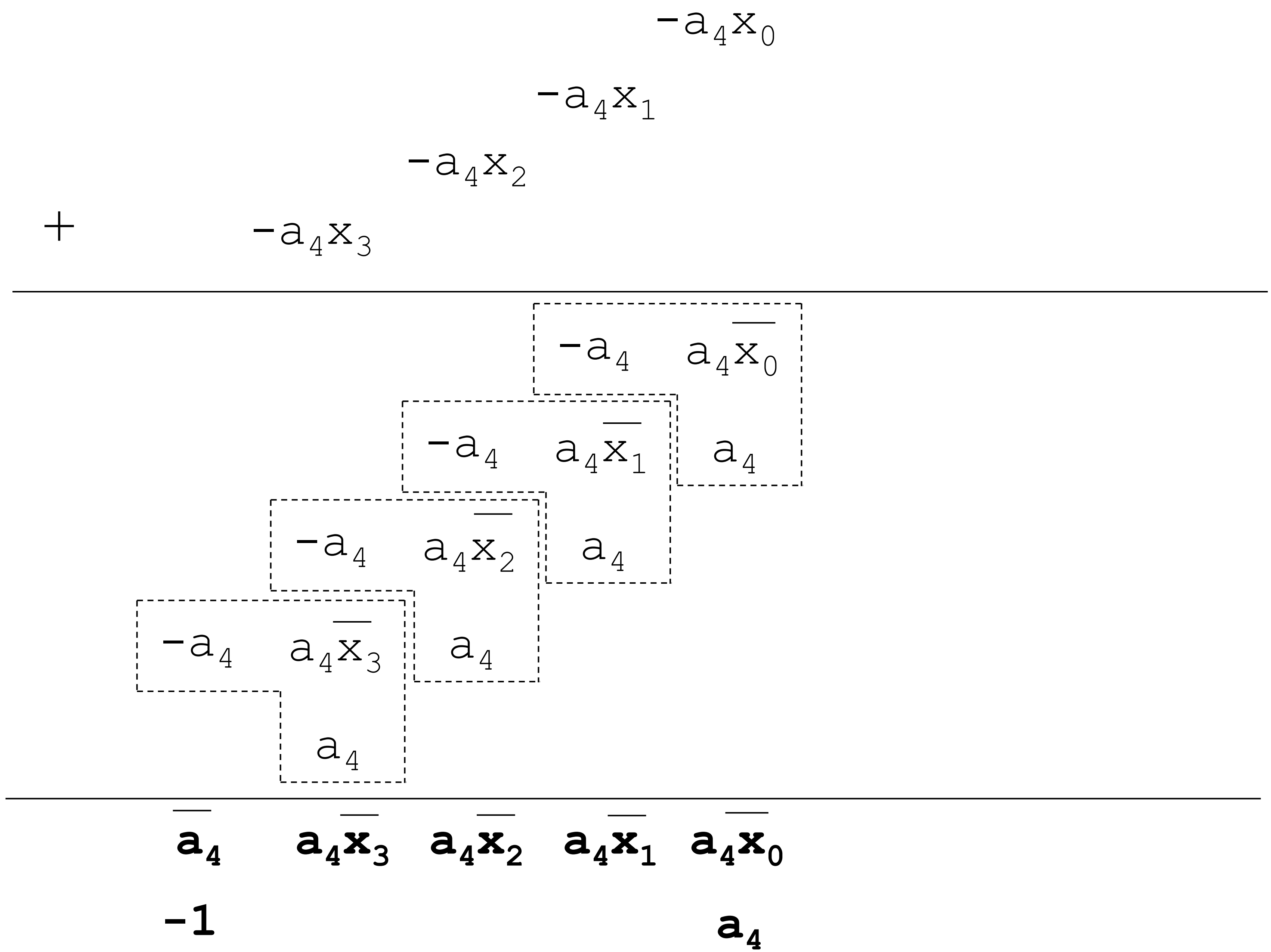
$$- a_j x_i = - a_j (1 - \overline{x_i}) = a_j \overline{x_i} - a_j = a_j \overline{x_i} + a_j - 2 a_j$$

$$- a_j x_i = - (1- \overline{a_j}) x_i = \overline{a_j} x_i - x_i = \overline{a_j} x_i + x_i - 2 x_i$$

$$- a_j x_i = - (1- \overline{a_j x_i}) = \overline{a_j x_i} - 1 = \overline{a_j x_i} + 1 - 2$$

$$-a_j = - (1 - \overline{a_j}) = \overline{a_j} - 1 = \overline{a_j} + 1 - 2$$

$$-x_i = - (1 - \overline{x_i}) = \overline{x_i} - 1 = \overline{x_i} + 1 - 2$$



$-a_4x_0$
$-a_4x_1$
$-a_4x_2$
$+$
$-a_4x_3$
$-a_4$ $a_4\overline{x_0}$
$-a_4$ $a_4\overline{x_1}$ a_4
$-a_4$ $a_4\overline{x_2}$ a_4
$-a_4$ $a_4\overline{x_3}$ a_4
a_4
$\overline{a_4}$ $a_4\overline{x_3}$ $a_4\overline{x_2}$ $a_4\overline{x_1}$ $a_4\overline{x_0}$
-1 a_4

$+ \quad -a_3x_4 \quad -a_2x_4 \quad -a_1x_4 \quad -a_0x_4$

$-x_4 \quad \overline{a_0}x_4$

x_4

$-x_4 \quad \overline{a_1}x_4$

x_4

$-x_4 \quad \overline{a_2}x_4$

x_4

$-x_4 \quad \overline{a_3}x_4$

x_4

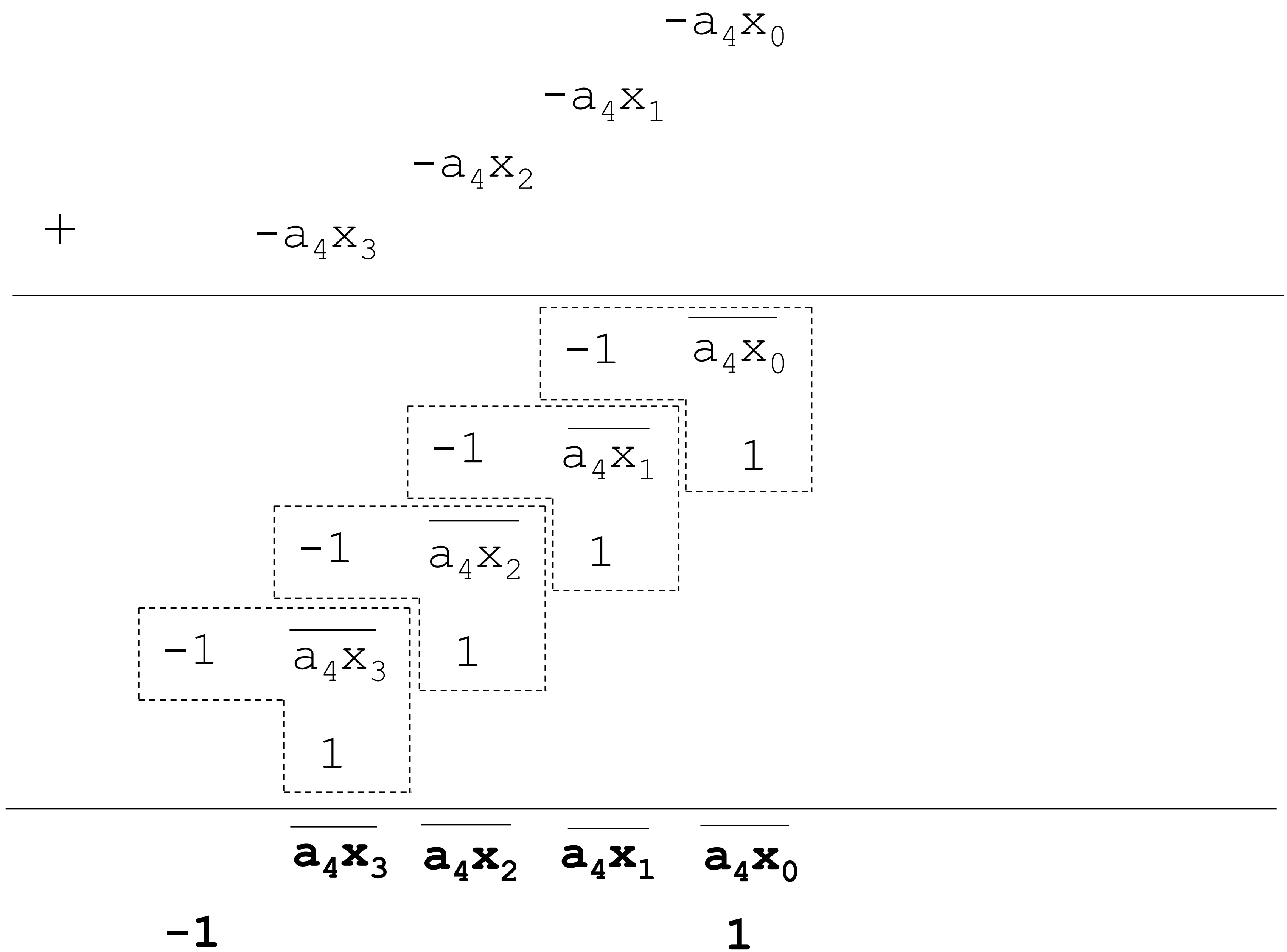
$\mathbf{\overline{x_4} \quad \overline{a_3}x_4 \quad \overline{a_2}x_4 \quad \overline{a_1}x_4 \quad \overline{a_0}x_4}$

$\mathbf{-1} \qquad \qquad \mathbf{x_4}$

2^9	2^8	2^7	2^6	2^5	2^4
	$\overline{a_4}$	$a_4\overline{x_3}$	$a_4\overline{x_2}$	$a_4\overline{x_1}$	$a_4\overline{x_0}$
	-1				a_4
	$\overline{x_4}$	$\overline{a_3}x_4$	$\overline{a_2}x_4$	$\overline{a_1}x_4$	$\overline{a_0}x_4$
	-1				x_4
---	---	---	---	---	---
-1	$\overline{a_4}$	$a_4\overline{x_3}$	$a_4\overline{x_2}$	$a_4\overline{x_1}$	$a_4\overline{x_0}$
	$\overline{x_4}$	$\overline{a_3}x_4$	$\overline{a_2}x_4$	$\overline{a_1}x_4$	$\overline{a_0}x_4$
					a_4
					x_4
---	---	---	---	---	---
1	$\overline{a_4}$	$a_4\overline{x_3}$	$a_4\overline{x_2}$	$a_4\overline{x_1}$	$a_4\overline{x_0}$
	$\overline{x_4}$	$\overline{a_3}x_4$	$\overline{a_2}x_4$	$\overline{a_1}x_4$	$\overline{a_0}x_4$
					a_4
-2^9					x_4

Baugh-Wooley 2's Complement Multiplier

						$-a_4$	a_3	a_2	a_1	a_0
					x	$-x_4$	x_3	x_2	x_1	x_0
						$a_4\overline{x_0}$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
		+			$a_4\overline{x_1}$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$a_4\overline{x_2}$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$a_4\overline{x_3}$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$\overline{a_3}x_4$	$\overline{a_2}x_4$	$\overline{a_1}x_4$	$\overline{a_0}x_4$				
		$\overline{a_4}$				a_4				
	1	$\overline{x_4}$				x_4				
	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	-2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0


$-a_4x_0$
$-a_4x_1$
$-a_4x_2$
$+$
$-a_4x_3$
-1 $\overline{a_4x_0}$ 1
-1 $\overline{a_4x_1}$ 1
-1 $\overline{a_4x_2}$ 1
-1 $\overline{a_4x_3}$ 1
$\overline{a_4x_3}$ $\overline{a_4x_2}$ $\overline{a_4x_1}$ $\overline{a_4x_0}$
-1 1

$$+ \quad -a_3x_4 \quad -a_2x_4 \quad -a_1x_4 \quad -a_0x_4$$

$$-1 \quad \overline{a_0x_4}$$

$$-1 \quad \overline{a_1x_4} \quad 1$$

$$-1 \quad \overline{a_2x_4} \quad 1$$

$$-1 \quad \overline{a_3x_4} \quad 1$$

$$1$$

$$\mathbf{\overline{a_3x_4} \quad \overline{a_2x_4} \quad \overline{a_1x_4} \quad \overline{a_0}x_4}$$

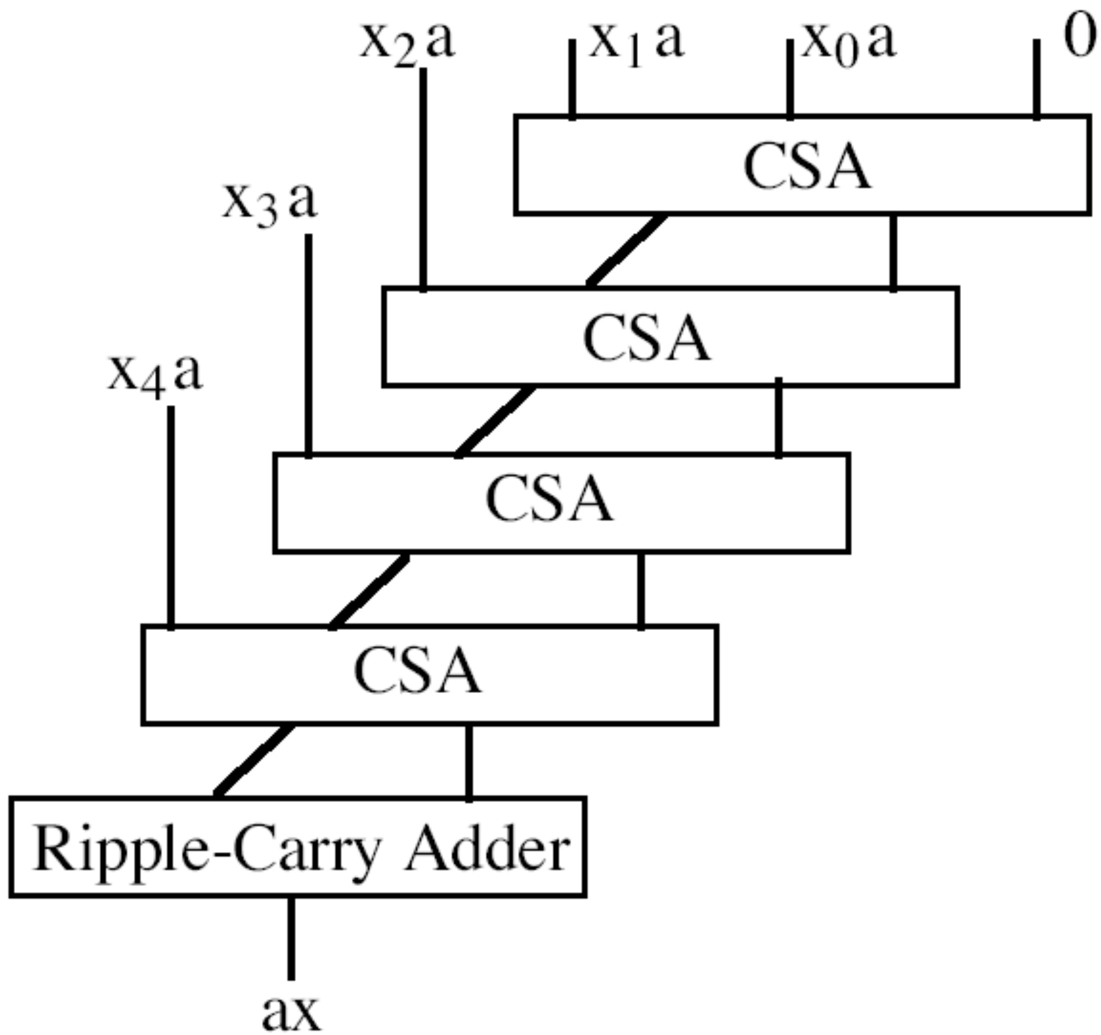
$$\mathbf{-1} \qquad\qquad\qquad\qquad \mathbf{1}$$

$\mathbf{2^9}$	2^8	2^7	2^6	2^5	2^4
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
	-1				1
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0}x_4$
	-1				1
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0}x_4$
-1				1	
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0}x_4$
1				1	
$\mathbf{-2^9}$					

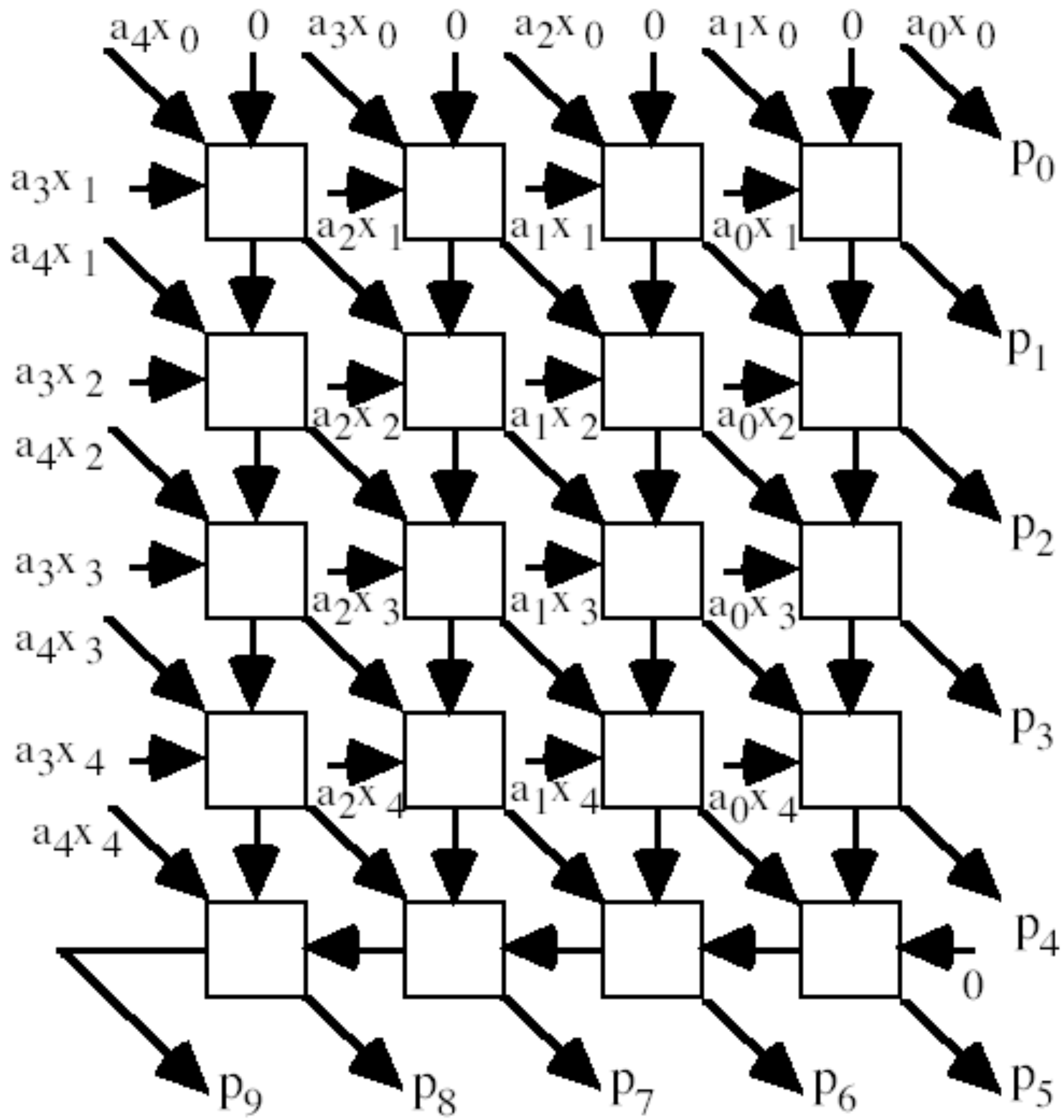
Modified Baugh-Wooley Multiplier

						$-a_4$	a_3	a_2	a_1	a_0
x						$-x_4$	x_3	x_2	x_1	x_0
						$\overline{a_4x_0}$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$\overline{a_4x_1}$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$\overline{a_4x_2}$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$\overline{a_4x_3}$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0x_4}$				
	1				1					
	$\mathbf{p_9}$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	$\mathbf{-2^9}$	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

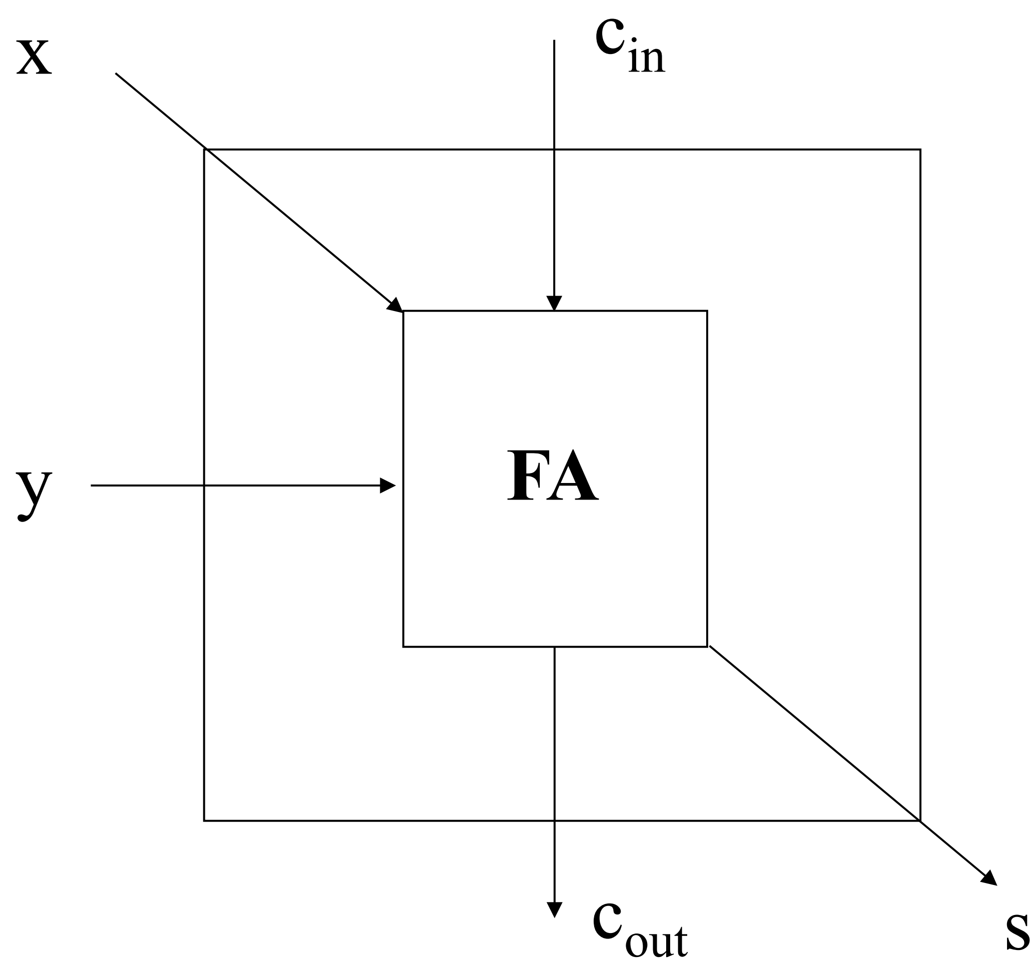
Basic array multiplier



5 x 5 Array Multiplier



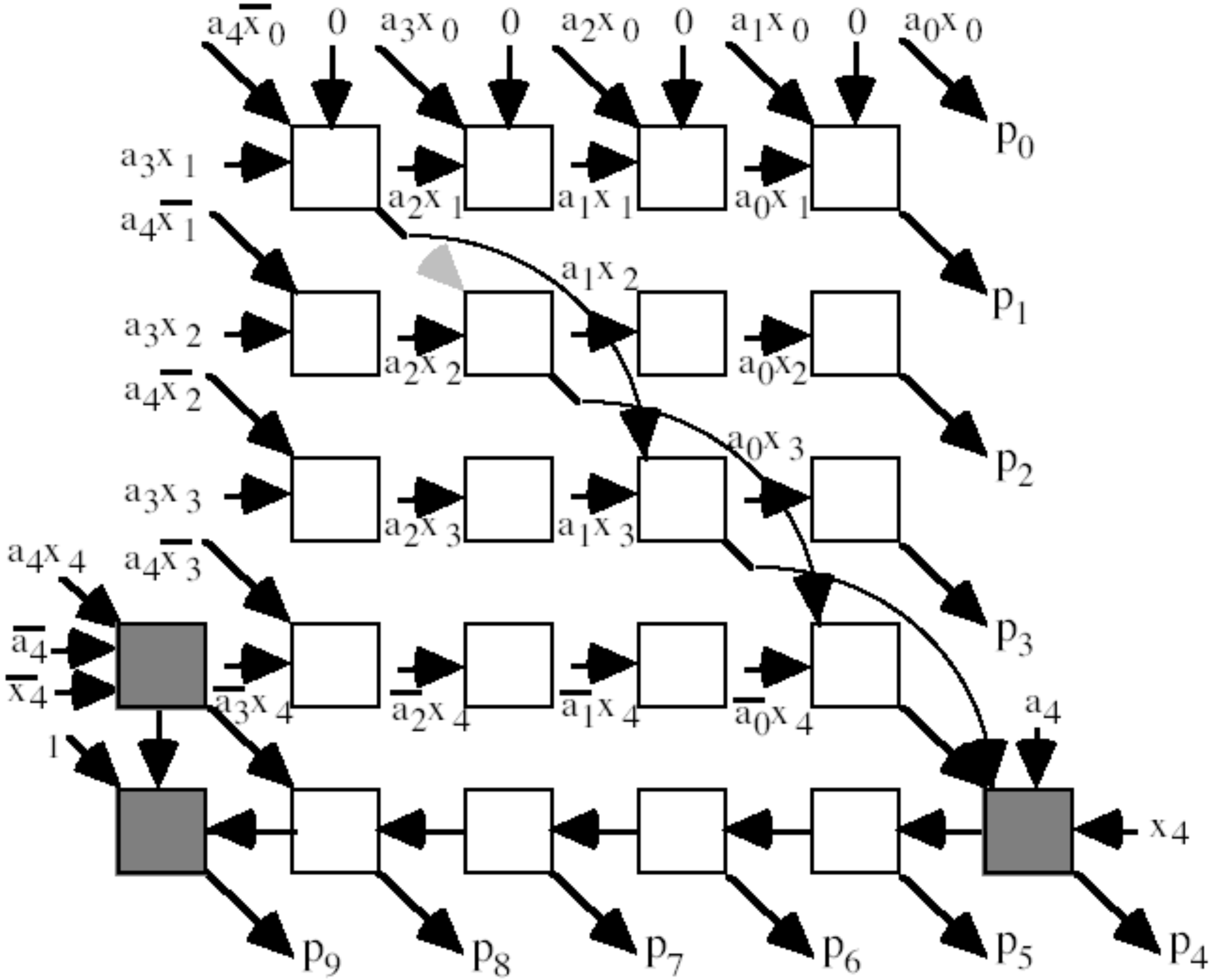
Array Multiplier - Basic Cell



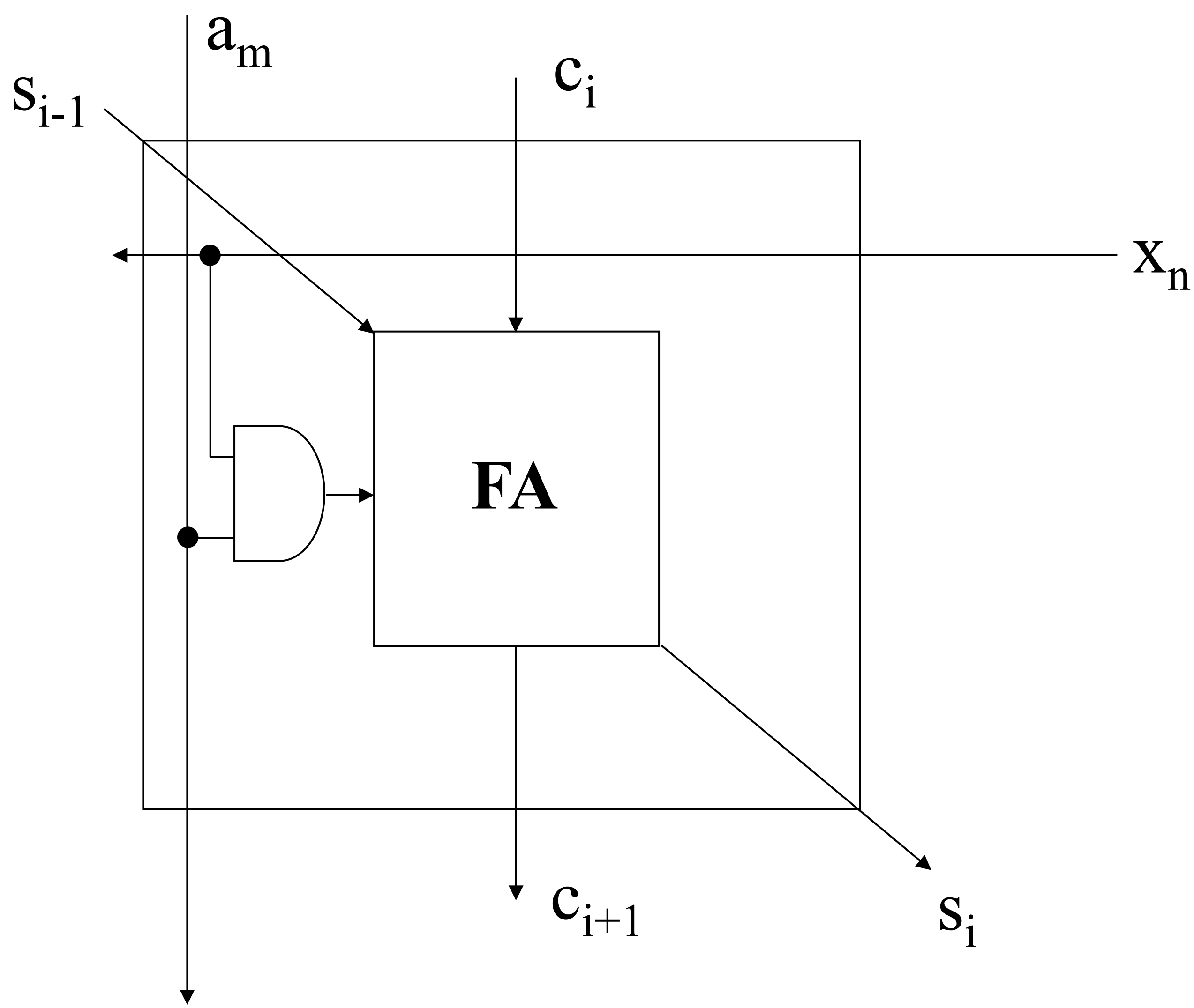
Baugh-Wooley 2's Complement Multiplier

						$-a_4$	a_3	a_2	a_1	a_0
x						$-x_4$	x_3	x_2	x_1	x_0
						$a_4\overline{x}_0$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$a_4\overline{x}_1$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$a_4\overline{x}_2$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$a_4\overline{x}_3$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$\overline{a}_3x_4$	$\overline{a}_2x_4$	$\overline{a}_1x_4$	$\overline{a}_0x_4$				
		$\overline{a}_4$				a_4				
	1	$\overline{x}_4$				x_4				
	$\mathbf{p_9}$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	$\mathbf{-2^9}$	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

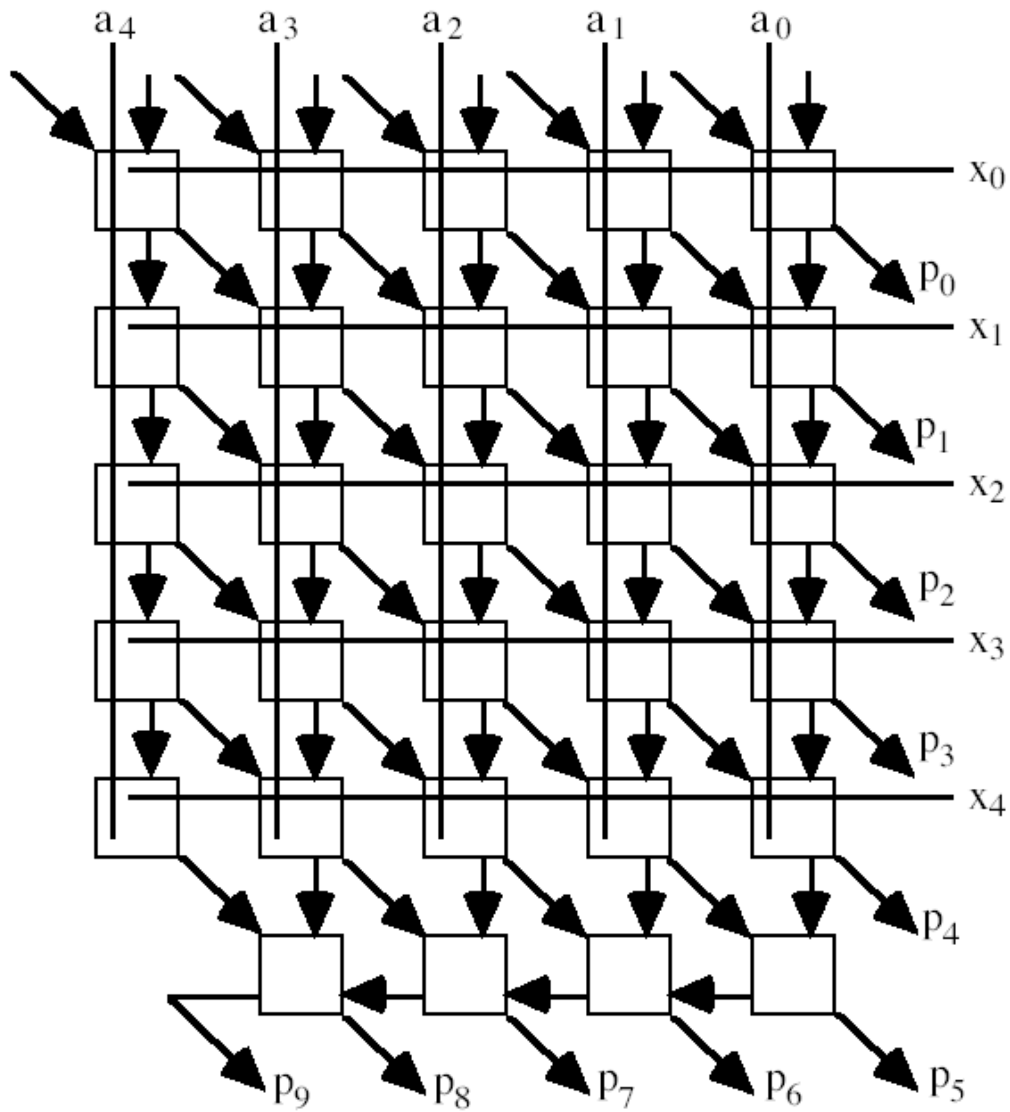
Modifications in a 5 x 5 multiplier



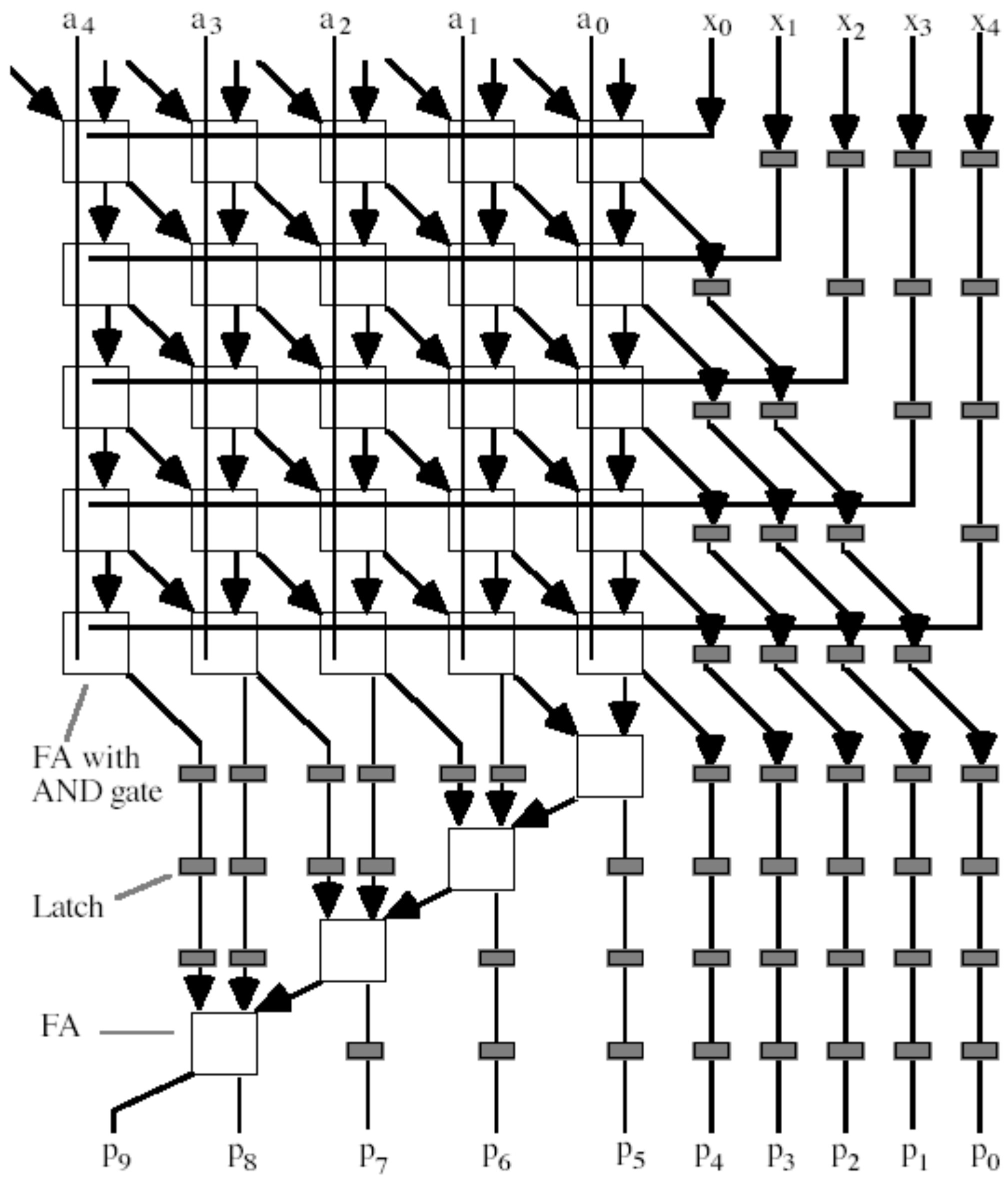
Array Multiplier – Modified Basic Cell



5 x 5 Array Multiplier with modified cells



Pipelined 5 x 5 Multiplier



Optimizations for Squaring (1)

Multiply x by x

						x_4	x_3	x_2	x_1	x_0
					×	x_4	x_3	x_2	x_1	x_0
						x_4x_0	x_3x_0	x_2x_0	x_1x_0	x_0x_0
				x_4x_1		x_3x_1	x_2x_1	x_1x_1	x_0x_1	Reduce
			x_4x_2	x_3x_2		x_2x_2	x_1x_2	x_0x_2	Move	to x_0
		x_4x_3	x_3x_3	x_2x_3		x_1x_3	x_0x_3		to next	
	x_4x_4	x_3x_4	x_2x_4	x_1x_4		x_0x_4			column	
p_9	p_8	p_7	p_6	p_5		p_4	p_3	p_2	p_1	p_0

Reduce the bit matrix

						x_4	x_3	x_2	x_1	x_0
					×	x_4	x_3	x_2	x_1	x_0
	x_4x_3	x_4x_2	x_4x_1	x_4x_0		x_3x_0	x_2x_0	x_1x_0	-	x_0
	x_4		x_3x_2	x_3x_1		x_2x_1		x_1		
			x_3			x_2				
p_9	p_8	p_7	p_6	p_5		p_4	p_3	p_2	0	x_0

Optimizations for Squaring (2)

$$
\begin{array}{r}
x_i\,x_j \\
x_j\,x_i \\
\hline
x_i\,x_j \quad\quad
\end{array}
$$

$$x_i\,x_j + x_i\,x_j = 2\,x_i\,x_j$$

$$x_i\,x_i = x_i$$

$$
\begin{array}{r}
x_i\,x_j \\
x_i \\
\hline
x_i\,x_j \quad x_i\,\overline{x_j}
\end{array}
$$

$$x_i\,x_j + x_i = 2\,x_i\,x_j - x_i\,x_j + x_i =$$

$$= 2\,x_i\,x_j + x_i\,(1-x_j) =$$

$$= 2\,x_i\,x_j + x_i\,\overline{x_j}$$

Squaring Using Look-Up Tables

for relatively small values k

input=a	output=a^2
0	0
1	1
2	4
3	9
4	16
	. . .
i	i^2
	. . .
2^k-1	$(2^k-1)^2$

2^k words 2k-bit each

Multiplication Using Squaring

$$a \cdot x = \frac{(a+x)^2 - (a-x)^2}{4}$$