Partial Fraction Expansion
for
Complex Conjugate Poles

In many real applications, a transfer function will have one or more pairs of complex conjugate poles, in addition to one or more real poles. If the complex poles have real parts equal to zero, then the poles are on the $j\omega$ axis and correspond to pure sinusoids. If the real parts of the complex poles are non-zero, then the poles are in the left-half of the $s$-plane if $\text{Re}[p] < 0$, and they are in the right-half of the $s$-plane if $\text{Re}[p] > 0$. Left-half plane complex poles correspond to sinusoids multiplied by decaying exponentials, and right-half plane complex poles correspond to sinusoids multiplied by growing exponentials. The relevant Laplace transform pairs for doing partial fraction expansion are given below.

\[
\mathcal{L} \left[ \sin (\omega_0 t) u(t) \right] = \frac{\omega_0}{s^2 + \omega_0^2}
\]

\[
\mathcal{L} \left[ \cos (\omega_0 t) u(t) \right] = \frac{s}{s^2 + \omega_0^2}
\]

\[
\mathcal{L} \left[ e^{-\alpha t} \sin (\omega_0 t) u(t) \right] = \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}
\]

\[
\mathcal{L} \left[ e^{-\alpha t} \cos (\omega_0 t) u(t) \right] = \frac{(s + \alpha)}{(s + \alpha)^2 + \omega_0^2}
\]

Various methods exist for finding the coefficients for the complex poles using partial fraction expansion. The coefficients for the real poles—distinct or repeated—are found in the usual ways for those poles. A simple example will demonstrate finding the coefficients for the complex poles.

Example 1

The transform of a signal is given by

\[
F(s) = \frac{N(s)}{D(s)} = \frac{s + 2}{(s^2 + 4s + 13)(s + 3)} = \frac{s + 2}{(s + 2 - j3)(s + 2 + j3)(s + 3)}
\]

This transform will be expanded, keeping the term involving the complex poles in its quadratic form.

\[
F(s) = \frac{s + 2}{(s^2 + 4s + 13)(s + 3)} = \frac{A_1 s + A_2}{(s^2 + 4s + 13)} + \frac{A_3}{(s + 3)}
\]

The coefficient $A_3$ can be found in the usual way.

\[
A_3 = (s + 3) F(s) \big|_{s=-3} = \frac{s + 2}{(s^2 + 4s + 13)} \bigg|_{s=-3} = \frac{-3 + 2}{9 - 12 + 13} = \frac{-1}{10} = -0.1
\]

If $F(s)$ had additional real poles, their coefficients would also be found at this time.

In order to determine the coefficients $A_1$ and $A_2$, the expansion on the right side of (2) is put over a common denominator using the coefficients already found, and the two forms of the numerator are equated for each power of $s$. This becomes the following.

\[
\frac{s + 2}{(s^2 + 4s + 13)(s + 3)} = \frac{(A_1 s + A_2)(s + 3) - 0.1 (s^2 + 4s + 13)}{(s^2 + 4s + 13)(s + 3)}
\]

\[
= \frac{(A_1 - 0.1) s^2 + (A_2 + 3A_1 - 0.4) s + (3A_2 - 1.3)}{(s^2 + 4s + 13)(s + 3)}
\]

Equating the two numerator polynomials in (5), we get the following results.
The result from the \( s^0 \) row is obviously the same as that from the \( s^1 \) row, which it must be for a correct result.

The transform can now be written as

\[
F(s) = \frac{0.1s + 1.1}{(s^2 + 4s + 13)} - \frac{0.1}{(s + 3)}
\]  
(6)

In order to go back to the time domain, the term in (6) must be put into one of the standard forms in the table at the beginning of these notes. Since the complex poles have non-zero real parts, the last two rows of that table are the relevant ones. Looking only at the quadratic term of \( F(s) \), we require

\[
\frac{0.1s + 1.1}{(s^2 + 4s + 13)} = \frac{0.1s + 1.1}{(s + 2)^2 + 3^2} = K_1 \cdot \frac{(s + 2)}{(s + 2)^2 + 3^2} + K_2 \cdot \frac{3}{(s + 2)^2 + 3^2}
\]  
(7)

which would give the time-domain expression for the complex poles

\[
K_1 e^{-2t} \cos(3t) + K_2 e^{-2t} \sin(3t)
\]  
(8)

The numerators in (7) can be equated to determine \( K_1 \) and \( K_2 \). The results are

\[
s^1: \quad 0.1 = K_1 \quad \Rightarrow \quad K_1 = 0.1
\]

\[
s^0: \quad 1.1 = 2K_1 + 3K_2 \quad \Rightarrow \quad K_2 = (1.1 - 2K_1)/3 = 0.3
\]

The final result is

\[
F(s) = 0.1 \cdot \frac{(s + 2)}{(s + 2)^2 + 3^2} + 0.3 \cdot \frac{3}{(s + 2)^2 + 3^2} - 0.1 \cdot \frac{1}{(s + 3)}
\]  
(9)

which yields the time-domain expression

\[
f(t) = [0.1 e^{-2t} \cos(3t) + 0.3 e^{-2t} \sin(3t) - 0.1 e^{-3t}] \cdot u(t)
\]  
(10)

Alternate Method

A pair of complex conjugate poles are distinct poles since they are not equal to one another. Therefore, the normal method for distinct real poles can be used also for complex conjugate poles. In this case, the coefficients for each of the poles will be complex, and the coefficients will be complex conjugates. Let the two complex poles be \( p_1 = \alpha + j\beta \) and \( p_1^* = \alpha - j\beta \). A transform with these poles can be written as

\[
F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s - p_1)(s - p_1^*)} D(s) = \frac{k_1}{(s - p_1)} + \frac{k_1^*}{(s - p_1^*)} + \frac{N(s)}{D(s)}
\]  
(11)

where \( D(s) \) is that part of the denominator of \( F(s) \) not involving the complex conjugate poles, and \( N(s) \) is a polynomial of lesser degree than \( D(s) \). The coefficient \( k_1 \) can be found by multiplying both sides of (11) by \( (s - p_1) \) and evaluating the result at \( s = p_1 = \alpha + j\beta \). The resulting coefficient—which will be complex—is

\[
k_1 = (s - p_1) \mid F(s)_{s=p_1} = a + jb
\]  
(12)

The other coefficient for the complex poles is \( k_1^* = a - jb \). The inverse transform for these complex poles is

\[
L^{-1} \left[ \frac{k_1}{(s - p_1)} + \frac{k_1^*}{(s - p_1^*)} \right] = k_1 e^{p_1 t} + k_1^* e^{p_1^* t} = (a + jb) e^{\alpha t} e^{j\beta t} + (a - jb) e^{\alpha t} e^{-j\beta t}
\]  
(13)

A property of any pair of complex conjugate numbers \( z_1 \) and \( z_1^* \) is
where the coefficient is found from

\[ k_1 = \frac{s + 2}{(s + 2 - j3)(s + 3)} \left| _{s = -2 + j3} \right. = \frac{-2 + j3 + 2}{(s + 2 - j3)(s + 3)} = \frac{j3}{(j6)(1 + j3)} = 0.05 - j0.15 \]

so the complex conjugate coefficient is \( k_1^* = 0.05 + j0.15 \), and

\[ |k_1| = \sqrt{0.05^2 + (-0.15)^2} = 0.1581, \quad \angle k_1 = \tan^{-1} \left( \frac{-0.15}{0.03} \right) = -1.249 \text{ rad} \Rightarrow -71.6^\circ \]

The time domain signal is

\[ f(t) = [0.3162e^{-2t} \cos (3t - 1.249) - 0.1e^{-3t}] \cdot u(t) \] (21)

**Another Alternate Form**

The method used above, namely evaluating the complex coefficient in the usual way for distinct poles resulted in a time-domain solution involving the cosine function with a phase angle. That method of evaluation can also be used to produce the solution with both cosine and sine functions in the following way.

\[
\mathcal{L}^{-1} \left[ \frac{k_1}{s - p_1} + \frac{k_1^*}{s - p_1^*} \right] = \mathcal{L}^{-1} \left[ \frac{a + jb}{s - \alpha - j\beta} + \frac{a - jb}{s - \alpha + j\beta} \right] = \mathcal{L}^{-1} \left[ \frac{(a + jb)(s - \alpha + j\beta) + (a - jb)(s - \alpha - j\beta)}{(s - \alpha - j\beta)(s - \alpha + j\beta)} \right]
\]

\[ = \mathcal{L}^{-1} \left[ \frac{2a(s - \alpha) - 2b\beta}{(s - \alpha)^2 + \beta^2} \right] \]

\[ = 2a \mathcal{L}^{-1} \left[ \frac{(s - \alpha)}{(s - \alpha)^2 + \beta^2} \right] - 2b \mathcal{L}^{-1} \left[ \frac{\beta}{(s - \alpha)^2 + \beta^2} \right] \]

\[ = 2ae^{\alpha t} \cos (\beta t) - 2be^{\alpha t} \sin (\beta t) \] (26)

Compare this with Example 1 with \( a = 0.05, b = -0.15, \alpha = -2, \) and \( \beta = 3. \)