Full-State Feedback Control Design Examples

A. Example 1

The open-loop system is described by the following state space model.

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
\]

(1)

\[
A = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0
\]

(2)

Obviously, the model is second order and represents a single-input, single-output (SISO) system, that is, \(n = 2\) and \(r = m = 1\). The eigenvalues of the open-loop system are \(\lambda_{OL} = \{2.5 \pm j1.66\}\). Since the open-loop system is unstable, some form of feedback is required to produce a stable system. In this example, full-state feedback will be used. The controllability matrix for the open-loop system is

\[
P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix}, \quad \text{Rank}(P) = r_P = 2 = n
\]

(3)

Since the rank of \(P\) is equal to \(n\), the system is completely controllable. Therefore, the closed-loop eigenvalues can be placed at arbitrary places in the complex plane as long as if one of the eigenvalues is complex, its complex conjugate is also an eigenvalue. For this example, the closed-loop eigenvalues will be placed at \(\lambda_{CL} = \{-1, -2\}\). Although this choice of closed-loop eigenvalues may not give satisfactory performance, they are asymptotically stable and will serve for this example.

With full-state feedback, \(u(t) = -Kx(t) + v(t)\), and the closed-loop state equations are

\[
\dot{x}(t) = (A - BK)x(t) + Bv(t)
\]

(4)

The closed-loop eigenvalues and eigenvectors are related by

\[
[A - BK] \psi_i = \lambda_i \psi_i \quad \Rightarrow \quad [\lambda_i I - A + BK] \psi_i = 0
\]

(5)

Since the \(r \times n\) feedback gain matrix \(K\) is unknown, Eqn. (5) can be rewritten as shown below which will lead to a procedure for computing the value for \(K\).

\[
[\lambda_i I - A \ B K] \begin{bmatrix} \psi_i \\ \psi_i \end{bmatrix} = 0 \quad \Rightarrow \quad [\lambda_i I - A \ B] \begin{bmatrix} \psi_i \\ K\psi_i \end{bmatrix} = 0 \quad \Rightarrow \quad [\lambda_i I - A \ B] \xi_i = 0
\]

(6)

where \(\psi_i\) is the \(n\)-dimensional eigenvector associated with eigenvalue \(\lambda_i\), and \(\xi_i\) is a vector of dimension \(n + r\) (3 in this example).

For each desired closed-loop eigenvalue \(\lambda_i\), the matrix \([\lambda_i I - A \ B]\) will be formed, and the Row-Reduced Echelon (RRE) technique will be used to find the vector \(\xi_i\). Once that is done for each \(\lambda_i\), the gain matrix \(K\) can be computed by partitioning the \(\xi_i\) into \(\psi_i\) and \(K\psi_i\). The procedure is described in detail for this example.

For \(\lambda = -1\), the steps in the process are:

\[
[\begin{array}{cc} -1 & 1 \\ 0 & 0 \end{array}] \xi_1 = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ -1 & -4 \end{bmatrix} \xi_1 = 0 \Rightarrow \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix} \xi_1 = 0
\]

(7)

\[
[\begin{array}{cc} -2 & 3 \\ 0 & 5 \end{array}] \xi_2 = 0 \Rightarrow \begin{bmatrix} -4 & 3 \\ -1 & -5 \end{bmatrix} \xi_2 = 0 \Rightarrow \begin{bmatrix} 1 & -1/4 \\ 1 & 1 \end{bmatrix} \xi_2 = 0
\]

(8)

The last matrix in (8) is in the RRE form. The first row of that matrix shows that the first element in \(\xi_1\) equals 0.6667 times the third element, and the second row of the matrix shows that the second element in \(\xi_1\) equals \(-0.2667\) times the third element. The third element of \(\xi_1\), which corresponds to \(K\psi_1\), is arbitrary. That value will be set to 1, so \(\xi_1 = [0.6667 \ -0.2667 \ 1]^T\).

For \(\lambda = -2\), the same steps will be followed.

\[
[\begin{array}{cc} 1 & 1/4 \\ 0 & 1 \end{array}] \xi_2 = 0 \Rightarrow \begin{bmatrix} -3/4 & 3 \\ -1 & -5 \end{bmatrix} \xi_2 = 0 \Rightarrow \begin{bmatrix} 1 & -1/4 \\ 5 & 1 \end{bmatrix} \xi_2 = 0
\]

(9)

\[
[\begin{array}{cc} 1 & 1/4 \\ 0 & 1 \end{array}] \xi_2 = 0 \Rightarrow \begin{bmatrix} -3/4 & 3 \\ -1 & -5 \end{bmatrix} \xi_2 = 0 \Rightarrow \begin{bmatrix} 1 & -0.0870 \\ 0 & 0.2174 \end{bmatrix} \xi_2 = 0
\]

(10)
The last matrix in (10) is in the RRE form. The first row of that matrix shows that the first element in \( \xi_2 \) equals 0.087 times the third element, and the second row of the matrix shows that the second element in \( \xi_2 \) equals -0.2174 times the third element. The third element of \( \xi_2 \), which corresponds to \( K \psi_2 \), is arbitrary. That value will be set to 1, so \( \xi_2 = \begin{bmatrix} 0.087 & -0.2174 & 1 \end{bmatrix}^T \).

The columns of the modal matrix \( M \) of eigenvectors are the first \( n = 2 \) rows of the vectors \( \xi_1 \) and \( \xi_2 \). The third elements of those two vectors form a matrix \( Q \) (not to be confused with the observability matrix). For this example, \( M \) and \( Q \) are

\[
M = \begin{bmatrix} 0.0667 & 0.0870 \\ -0.2667 & -0.2174 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \end{bmatrix}
\]  (11)

Since \( Q = \begin{bmatrix} K \psi_1 & K \psi_2 \end{bmatrix} \) and \( M = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \), it should be clear that \( Q = KM \), and the gain matrix can be solved from

\[
K = QM^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0.0667 & 0.0870 \\ -0.2667 & -0.2174 \end{bmatrix}^{-1} = \begin{bmatrix} 5.6667 & -2.3333 \end{bmatrix}
\]  (12)

The closed-loop matrix \( A_{CL} = A - BK \) and its eigenvalues are given by

\[
A_{CL} = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5.6667 & -2.3333 \end{bmatrix} = \begin{bmatrix} -3.6667 & -0.6667 \\ 6.6667 & 0.6667 \end{bmatrix}, \quad \lambda_{CL} = \{-1, -2\}
\]  (13)

The final closed-loop system is

\[
\dot{x}(t) = \begin{bmatrix} -3.6667 & -0.6667 \\ 6.6667 & 0.6667 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} v(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\]  (14)

### B. Example 2

For this example, the open-loop system model is the same as in the first example. The only change is in the desired closed-loop eigenvalues. Now the desired eigenvalues are \( \lambda_{CL} = \{-2, -2\} \). Since the eigenvalues are repeated, there is a possibility that there will be one eigenvector and one generalized eigenvector for this system. If that is the case, the RRE procedure will need to slightly modified.

The matrix \( \begin{bmatrix} \lambda I - A & B \end{bmatrix} \) is given by

\[
\begin{bmatrix} \lambda I - A & B \end{bmatrix} = \begin{bmatrix} \lambda - 2 & 3 & 1 \\ -1 & \lambda - 3 & -1 \end{bmatrix}
\]  (15)

The rank of \( \begin{bmatrix} \lambda I - A & B \end{bmatrix} \) equals 2 regardless of the value of \( \lambda \). Since the matrix has 3 columns and the eigenvalue's multiplicity is 2, the degeneracy is \( q = 3 - 2 = 1 \). Therefore, for the repeated eigenvalue \( \lambda = -2 \), there is one eigenvector and one generalized eigenvector (the simple degeneracy case). The eigenvector can be found using the same procedure as in Example 1. Since the eigenvalue in this example is the same as the second eigenvalue in the first example, the same eigenvector will be obtained. Thus, \( \xi_1 = \begin{bmatrix} 0.087 & -0.2174 & 1 \end{bmatrix}^T \).

The generalized eigenvector will be found using the chain rule discussed in Chapter 7 of the text. The eigenvalue/eigenvector expression in Eqn. (5) with the repeated eigenvalue becomes

\[
[\lambda_2 I - A + BK] \psi_2 = -\psi_1
\]  (16)

The RRE method can be applied to this equation as before, but it must also be applied to the vector \(-\psi_1\). An easy way to accomplish this is to include that vector with the \([\lambda_2 I - A + BK] \) matrix and perform RRE on the following matrix.

\[
\begin{bmatrix} \lambda_2 I - A & B & -\psi_1 \end{bmatrix}
\]  (17)

That matrix and the result of the RRE process are shown below.

\[
\begin{bmatrix} -2I - A & B & -\psi_1 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 1 & -0.087 \\ -1 & -5 & -1 & 0.2174 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -0.087 & -0.0095 \\ 0 & 1 & 0.2174 & -0.0416 \end{bmatrix}
\]  (18)

As before, the third element in the \( \xi_2 \) vector will be arbitrary. Letting that value be \( \alpha_2 \) for the moment, the first row of the final RRE matrix shows that the first element of \( \xi_2 \) equals \( 0.087\alpha_2 + 0.0095 \), and the second element of \( \xi_2 \) equals \(-0.2174\alpha_2 + 0.0416 \). The following vectors show the resulting \( \xi_2 \) for several different values of \( \alpha_2 \).

Any one of the vectors shown in (19) used along with the first vector $\xi_1$ for this example will produce the same $K$ matrix. For example, if $\alpha_2 = -2$ is used, the gain matrix is

$$K = QM^{-1} = \begin{bmatrix} 1 & -2 \\ 0.087 & -0.2174 \\ -0.1645 & 0.4764 \end{bmatrix}^{-1} = \begin{bmatrix} 7.3333 & -1.6667 \end{bmatrix}$$

(20)

and if $\alpha_2 = 13$ is used, the same gain matrix is obtained.

$$K = QM^{-1} = \begin{bmatrix} 1 & 13 \\ 0.087 & -0.2174 \\ -0.1645 & 0.4764 \end{bmatrix}^{-1} = \begin{bmatrix} 7.3333 & -1.6667 \end{bmatrix}$$

(21)

The closed-loop matrix $A_{CL} = A - BK$ and its eigenvalues are given by

$$A_{CL} = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 7.3333 & -1.6667 \end{bmatrix} = \begin{bmatrix} -5.3333 & -1.3333 \\ 8.3333 & 1.3333 \end{bmatrix}, \quad \lambda_{CL} = \{-2, -2\}$$

(22)

and the final closed-loop system is

$$\dot{x}(t) = \begin{bmatrix} -5.3333 & -1.3333 \\ 8.3333 & 1.3333 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} v(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

(23)