Multivariable Control: Placement of Eigenvectors

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Presentation Outline

- Problem Statement
- Motivating Example
- Why Eigenvectors Are Important
- How Eigenvectors Can Be Found
- Which Eigenvectors Might Be Selected
- Example Continued
- Conclusions
Problem Statement

- Given either state-space description for a system:
  \[ \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = -Lx(t) \]
  \[ x(k+1) = \Phi x(k) + \Gamma u(k), \quad u(k) = -Lx(k) \]
- Choose feedback gain \( L \) so that:
  - Eigenvalues are placed at specified locations;
    \[ |\lambda I - A + BL| = 0 \text{ or } |\lambda I - \Phi + \Gamma L| = 0 \]
  - Good performance is achieved.
Problem Statement

- For a single-input system, $L$ is unique for a given set of eigenvalues.

- For a multi-input system, $L$ is **not** unique for a given set of eigenvalues.

- For the same eigenvalues, different $L$ matrices will produce:
  - Different eigenvectors;
  - Different performance.
Motivating Example

- Discretized system model with $T_{\text{samp}} = 0.05$ seconds:

$$
\Phi = \begin{bmatrix}
1.0010 & 0.0501 & 0.0012 \\
0.0049 & 1.0050 & 0.0489 \\
0.1954 & 0.2003 & 0.9562 \\
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
0.0500 & 0.0000 \\
0.0001 & 0.0012 \\
0.0049 & 0.0489 \\
\end{bmatrix}, \quad \lambda_{OL} = \begin{bmatrix} 0.9524 \\
0.9045 \\
1.1054 \end{bmatrix}
$$

$$
x(0) = [10 \quad 10 \quad 10]^T
$$

- Desired closed-loop eigenvalues are: $\lambda_{CL} = 0.9, 0.8, 0.7$

$$
L_1 = \begin{bmatrix}
2.020 & 1.002 & 0.024 \\
11.525 & 28.568 & 8.726 \\
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
-105.74 & -243.56 & -91.52 \\
142.48 & 323.76 & 121.34 \\
\end{bmatrix}
$$
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Motivating Example

- Eigenvalues are identical with each gain matrix.
- Settling times are similar for the two sets of responses.
- Some signal shapes are similar \((x_2, x_3, u_2)\).
- \(L_2\) produces much larger signal amplitudes than \(L_1\) for all variables.
- Quantitative comparison between the 2 responses:

\[
J = \sum_{k=0}^{80} \left[ x^T(k)Qx(k) + u^T(k)Ru(k) \right]
\]

\[
Q = I_3, \quad R = I_2
\]

\[
J_1 = 379,450, \quad J_2 = 84,243,000
\]
**Eigenvector Importance**

- If the following control law is used:

  \[
  u = -Lx, \quad A_{CL} = A - BL \quad \text{or} \quad \Phi_{CL} = \Phi - \Gamma L
  \]

- The closed-loop state equations are:

  \[
  x(k) = \Phi_{CL}^k x(0), \quad x(t) = e^{A_{CL}t} x(0)
  \]

- If the system has real and distinct closed-loop eigenvalues \(\lambda_i\):

  \[
  A_{CL} = T\Lambda T^{-1}, \quad \Lambda = \text{diag}[\lambda_1 \lambda_2 \ldots \lambda_n]
  \]

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**Eigenvector Importance**

- Right eigenvectors for the system are:
  \[ T = [v_1 : v_2 : \cdots : v_n], \quad [\lambda_i I - A_{CL}] v_i = 0 \text{ or } [\lambda_i I - \Phi_{CL}] v_i = 0 \]

- Left eigenvectors for the system are:
  \[ T^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}, \quad w_i^T [\lambda_i I - A_{CL}] = 0 \text{ or } w_i^T [\lambda_i I - \Phi_{CL}] = 0 \]

- The closed-loop system matrices are:
  \[ \Phi_{CL}^k = T \Lambda^k T^{-1}, \quad e^{A_{CL}t} = Te^{\Lambda t} T^{-1} \]
**Eigenvector Importance**

- Solutions to the closed-loop state equations are:

\[
x(k) = \sum_{i=1}^{n} v_i \lambda_i^k [w_i^T x(0)], \quad x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} [w_i^T x(0)]
\]

- The eigenvalue \( \lambda_i \) defines the mode of the system.

- Product of left eigenvector and initial conditions \( w_i^T x(0) \) determines the amplitude of the mode.

- Right eigenvector \( v_i \) determines the shape of the mode.
Finding Eigenvectors

- Each closed-loop right eigenvector satisfies

\[
[\lambda_i I - A + BL]v_i = 0 \quad \text{or} \quad [\lambda_i I - \Phi + \Gamma L]v_i = 0
\]

- Eigenvector \(v_i\) must be in the null space of \(\lambda_i I - A + BL\) or \(\lambda_i I - \Phi + \Gamma L\)
- Eigenvectors \(v_i\) and \(v_j\) must be linearly independent for \(i \neq j\).

- But there is a problem: feedback matrix \(L\) is unknown!
- Defining \(q_i = Lv_i\) yields

\[
\begin{bmatrix}
\lambda_i I - A & B \\
q_i
\end{bmatrix}
\begin{bmatrix}
v_i \\
\cdots
\end{bmatrix} = 0 \quad \text{or} \quad
\begin{bmatrix}
\lambda_i I - \Phi & \Gamma \\
q_i
\end{bmatrix}
\begin{bmatrix}
v_i \\
\cdots
\end{bmatrix} = 0
\]

\[v_i = n \times 1, \quad q_i = m \times 1\]
Finding Eigenvectors

- For each $\lambda_i$, the dimension of the null space $= m =$ number of control inputs.
- Singular value decomposition (SVD) can be used to find a basis for the null space.
- For the $n$ closed-loop eigenvalues $\lambda_i$

$$[q_1 : q_2 : \cdots : q_n] = L[v_1 : v_2 : \cdots : v_n]$$

$$Q = LT \Rightarrow L = QT^{-1}$$

- That matrix $L$ places eigenvalues and eigenvectors at values specified by $\lambda_i$ and $v_i$, respectively.
Finding Eigenvectors

- Singular value decomposition: for each $\lambda_i$
  
  $$S_i = [\lambda_i I - A : B] = U_i \Sigma_i (V_i)^T$$ or
  $$S_i = [\lambda_i I - \Phi : \Gamma] = U_i \Sigma_i (V_i)^T$$

  $$S_i = n \times (n + m), \quad U_i = n \times n$$
  $$\Sigma_i = n \times (n + m), \quad V_i = (n + m) \times (n + m)$$

- Last $m$ columns of $V_i$ are a basis for null space of $S_i$

  $$\begin{bmatrix} v_i \\ \cdots \\ q_i \end{bmatrix} = \sum_{k=1}^{m} \alpha_{i,k} V_{i,n+k}$$

- $v_i$ and $v_j$ must be linearly independent for $i \neq j$
Finding Eigenvectors

Procedure Summarized

- For each desired closed-loop eigenvalue $\lambda_i$:
  - Form the matrix $S_i$;
  - Form the SVD of $S_i$ to get $V_i$;
  - Choose weighting factors $\alpha_{i,k}$ to get $v_i$ and $q_i$.

- Compute the feedback gain matrix $L = QT^{-1}$
Selecting Eigenvectors

**IF** \( T \) (matrix of eigenvectors) is made to be diagonal by choice of \( v_i \):

**THEN**
- \( A_{CL} \) or \( \Phi_{CL} \) is diagonal;
- Response of each state variable is controlled by its own eigenvalue;
- Closed-loop system is diagonalized, even though the open-loop system was not diagonal.
Selecting Eigenvectors

- In general, $T$ cannot be made diagonal.

- Alternatives:
  - Choose the $v_i$ so that each eigenvector has as many entries as possible = 0 in off-diagonal positions in $T$, and linear independence is maintained.
  - Choose the $v_i$ so that the eigenvectors are as orthogonal as possible, and linear independence is maintained $\Rightarrow$ MATLAB© “place” function.

- Neither of these approaches is best in every application.
Example Continued

\[
V_{1,4-5} = \begin{bmatrix}
-0.284 & 0.000 \\
0.255 & 0.336 \\
-0.536 & -0.709 \\
-0.332 & 0.320 \\
-0.676 & 0.531
\end{bmatrix}, \quad V_{2,4-5} = \begin{bmatrix}
-0.216 & 0.000 \\
-0.057 & -0.089 \\
0.253 & 0.397 \\
-0.920 & -0.080 \\
-0.197 & 0.910
\end{bmatrix}, \quad V_{3,4-5} = \begin{bmatrix}
-0.161 & 0.000 \\
-0.015 & -0.030 \\
0.106 & 0.209 \\
-0.980 & -0.025 \\
-0.048 & 0.977
\end{bmatrix}
\]

\[
\lambda_1 = 0.9 \quad \lambda_2 = 0.8 \quad \lambda_3 = 0.7
\]

- \(L_1\): 1\textsuperscript{st} column for 0.9; 2\textsuperscript{nd} column for 0.8 and 0.7.
- \(L_2\): sum of 1\textsuperscript{st} and 2\textsuperscript{nd} columns for 0.9, 0.8, and 0.7.
- \(L_3\): 2\textsuperscript{nd} column for 0.9 and 0.8; \(\alpha_{3,k}\) chosen for 0.7 to make the entry in row 3 equal to 0.
- \(L_4\): uses MATLAB\textsuperscript{©} “place” function.
Example Continued

\[
\Phi_{CL_1} = \begin{bmatrix} 0.9000 & 0.0000 & 0.0000 \\ -0.0091 & 0.9706 & 0.0384 \\ -0.3781 & -1.2016 & 0.5294 \end{bmatrix}, \quad J_1 = 379,450
\]
\[\|L_1\|_2 = 32.058\]

\[
\Phi_{CL_2} = \begin{bmatrix} 6.2882 & 12.228 & 4.5771 \\ -0.1555 & 0.6408 & -0.0876 \\ -6.2539 & -14.438 & -4.5290 \end{bmatrix}, \quad J_2 = 84,243,000
\]
\[\|L_2\|_2 = 467.68\]

\[
\Phi_{CL_3} = \begin{bmatrix} 0.7000 & 0.0000 & 0.0000 \\ 0.0002 & 0.9903 & 0.0429 \\ -0.0003 & -0.4008 & 0.7097 \end{bmatrix}, \quad J_3 = 138,900
\]
\[\|L_3\|_2 = 13.875\]

\[
\Phi_{CL_4} = \begin{bmatrix} 0.7997 & -0.0287 & -0.0116 \\ 0.0001 & 0.9854 & 0.0405 \\ -0.0036 & -0.6006 & 0.6150 \end{bmatrix}, \quad J_4 = 169,400
\]
\[\|L_4\|_2 = 18.181\]
Example Continued

\[ L_1 = \begin{bmatrix} 2.0200 & 1.0220 & 0.0240 \\ 11.525 & 28.568 & 8.7260 \end{bmatrix} \]

\[ L_2 = \begin{bmatrix} -105.74 & -243.56 & -91.518 \\ 142.48 & 323.76 & 121.34 \end{bmatrix} \]

\[ L_3 = \begin{bmatrix} 6.0200 & 1.0220 & 0.0240 \\ 3.3989 & 12.192 & 5.0379 \end{bmatrix} \]

\[ L_4 = \begin{bmatrix} 4.0269 & 1.5757 & 0.2561 \\ 3.6665 & 16.219 & 6.9526 \end{bmatrix} \]
Comparison with Optimal LQR

- Weighting Matrices: $Q = I_3$, $R = I_2$

\[
\Phi_{CL_{LQR}} = \begin{bmatrix}
0.8352 & -0.1938 & -0.0831 \\
0.0025 & 1.0006 & 0.0472 \\
0.0945 & 0.0188 & 0.8858
\end{bmatrix}, \quad J_{LQR} = 71,879, \quad \|L_{LQR}\|_2 = 7.2451
\]

\[
L_{LQR} = \begin{bmatrix}
3.3158 & 4.8785 & 1.6856 \\
1.7306 & 3.2223 & 1.2713
\end{bmatrix}
\]

- Closed-loop eigenvalues:

\[
\lambda_{CL_{LQR}} = \begin{bmatrix}
0.9448 & 0.8884 + j0.0363 & 0.8884 - j0.0363
\end{bmatrix}
\]
Summary of Results

<table>
<thead>
<tr>
<th>Gain</th>
<th>$J$</th>
<th>$J/J_{LQR}$</th>
<th>$|L|_2$</th>
<th>$|L|<em>2/|L</em>{LQR}|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>379,450</td>
<td>5.28</td>
<td>32.058</td>
<td>4.42</td>
</tr>
<tr>
<td>$L_2$</td>
<td>84,243,000</td>
<td>1,172</td>
<td>467.68</td>
<td>64.6</td>
</tr>
<tr>
<td>$L_3$</td>
<td>138,900</td>
<td>1.93</td>
<td>13.875</td>
<td>1.92</td>
</tr>
<tr>
<td>$L_4$</td>
<td>169,400</td>
<td>2.36</td>
<td>18.181</td>
<td>2.51</td>
</tr>
<tr>
<td>$L_{LQR}$</td>
<td>71,879</td>
<td>1</td>
<td>7.2451</td>
<td>1</td>
</tr>
</tbody>
</table>

- Wide variation in performance is seen, even though the eigenvalues are fixed in location for $L_1$ - $L_4$.

- The optimal LQR solution does not guarantee the location of eigenvalues or “satisfactory” performance, only stability and optimality.
Conclusions

- Eigenvectors have a dramatic effect on closed-loop performance.
- For multi-input systems, eigenvectors should be chosen as well as eigenvalues, as far as possible.
- Eigenvectors should be as linearly independent as possible.
- The 2-norm of the feedback gain matrix seems to be a good measure of this independence.