Getting Started with your \LaTeX{\textsuperscript{\textregistered}} Technical Paper

First Step: Set up the preamble, which is the area between \texttt{\documentclass} and \texttt{\begin{document}}.

The preamble includes the definition of the document class with options,

\texttt{\documentclass[journal,onecolumn,twoside]{IEEEtran}}

Global style commands.

\texttt{\setlength{\parindent}{0pt}}

Packages that you want to include,

\texttt{\usepackage[pdftex]{graphicx}}
\texttt{\usepackage{amsmath,amssymb}}
\texttt{\usepackage{setspace}}
\texttt{\usepackage{subfigure}}
\texttt{\def\pr{{\rm P}}}\

Your own special features and definitions

\texttt{\begin{document}}

Paper Preamble

For the sample paper, the \texttt{preamble} is quite simple:

\texttt{\%\documentclass[journal,onecolumn,twoside]{IEEEtran}}
\texttt{\documentclass[10pt,twocolumn,twoside]{IEEEtran}}
\texttt{\usepackage[pdftex]{graphicx}}
\texttt{\usepackage{amsmath,amssymb}}
\texttt{\usepackage{setspace}}
\texttt{\usepackage{subfigure}}
\texttt{\def\pr{{\rm P}}}
\texttt{\begin{document}
Random Variables: An Overview
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(Invited Paper)

Abstract—This paper introduces the concept of a random variable, which is nothing more than a variable whose numeric value is determined by the outcome of an experiment. To describe the probabilities that are associated with these numeric values in a concise and conceptually useful manner, the probability distribution and probability density function are introduced. Then, the moment generating function is defined, and several examples are given. Finally, the concept of a correlation function and correlation matrices is introduced.
I. INTRODUCTION

The concept of a random variable is a simple one, and one that is important. Although perhaps sounding at first like something difficult, random variables are conceptually quite simple. Given a sample space \( \Omega \) corresponding to some random experiment, this sample space contains elementary events, \( \omega \in \Omega \), and when an experiment is performed, a specific elementary event (experimental outcome) is observed.

\[ P\{N = n\} = \frac{\lambda^n}{n!} e^{-\lambda} \quad n \geq 0 \]  

for some \( \lambda > 0 \).

Given this probability assignment for \( N \), it is then easy to find the probability of any event that is defined in terms of

\[ \text{Poisson random variable} \]

\begin{itemize}
    \item Let \( N \) be a variable that represents the number of \( \alpha \) particles that are counted over a given period of time. The ensemble for \( N \) is the set of non-negative integers
    \[ E_N = \{0, 1, 2, \ldots\} \]
    Since the number of outcomes is unknown until we actually make a count, then \( N \) is a random variable. In many cases, it is appropriate to model \( N \) as a Poisson random variable where\(^1\)
    \[ P\{N = n\} = \frac{\lambda^n}{n!} e^{-\lambda} \quad n \geq 0 \]
    for some \( \lambda > 0 \).
    \end{itemize}

\(^1\)Note that with this probability assignment it is assumed that the number of particles may be arbitrarily large and, in fact, approach infinity.
values of $N$. For example, the probability that the number of $\alpha$ particles is less than some number, $N_0$, may be found as follows. Since the event $\{N < N_0\}$ is the union of the events $\{N = k\}$ for $k = 0, 1, \ldots, N_0 - 1$,

$$\{N < N_0\} = \bigcup_{n=0}^{N_0-1} \{N = n\}$$

and since these events are mutually exclusive, then

$$P\{N < N_0\} = \sum_{n=0}^{N_0-1} P\{N = n\} = \sum_{n=0}^{N_0-1} \frac{\lambda^n}{n!} e^{-\lambda}$$

This last sum may be evaluated using the following

$$\sum_{n=0}^{k} \frac{\lambda^n}{n!} e^{-\lambda} = \frac{\Gamma(k+1, \lambda)}{k!}$$

where

$$\Gamma(k, \lambda) = \int_{\lambda}^{\infty} x^{k-1} e^{-x} dx$$

For example, the probability that the number of $\alpha$ particles is less than some number, $N_0$, may be found as follows.

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$$\bigcup_{\{N < N_0\}} = \bigcup_{\{N = n\}} \{N = n\}$$

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where

$$\Gamma(k, \lambda) = \int_{\lambda}^{\infty} x^{k-1} e^{-x} dx$$
Arrays with Invisible Delimiters

In order to express probability mass functions mathematically, we introduce the \textit{delta function},\footnote{In digital signal processing, $\delta[n]$ is referred to as the unit sample function.} which is defined as follows:

$$\delta[n] = \begin{cases} 1 & ; \ n = 0 \\ 0 & ; \ n \neq 0 \end{cases}$$

Shifted delta functions may be used to represent functions that have a value of one at other values of \(n\). For example, $\delta[n-1]$ is equal to one when $n = 1$ and equal to zero for all other values of \(n\). Therefore, for an integer-valued discrete random variable \(X\) with

$$\Pr[X = n] = p_X[n] \ ; \ -\infty < n < \infty$$

Fractions in Equations

Another example is the \textit{The Geometric Random Variable} that has an ensemble equal to the set of all positive integers

$$\mathcal{E}_X = \{1, 2, 3, \ldots\}$$

with a probability law given by

$$\Pr[N = k] = \left(\frac{1}{2}\right)^k \ ; \ k > 0$$

The probability mass function for this random variable is

$$p_N(n) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \delta[n - k]$$
Combinatorics and Phantoms

Another random variable that occurs frequently in applications is one that corresponds to the number of successes, $N$, in $n$ Bernoulli trials, with the probability of a success being equal to $p$. In this case, $N$ has a Binomial Distribution with

$$p_N(k) = P\{N = k\} = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n$$

where $\binom{n}{k}$ is the number of combinations of $n$ objects that are taken $k$ at a time, and is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Alternative notations include $C(n,k)$, $nC_k$, $C^n_k$, and $C^n_k$.

Substack Command

Interesting problems that are sometimes challenging to solve, are those such as

$$P\{N \text{ is odd}\} = \sum_{0 \leq n \leq \infty \atop n \text{ odd}} P\{N = n\}$$

\[ \text{}``p_{N}(k) = \text{pr}\{N = k\} \]
\[ = \dbinom{n}{k} p^k (1-p)^{n-k} \]
\[ \; \quad \text{0 \leq k \leq n} \]
\[ \text{where \$\dbinom{n}{k}\$ is the number of combinations of \$n\$ objects that are taken \$k\$ at a time, and is defined by} \]
\[ \dbinom{n}{k} = \frac{n!}{k!(n-k)!} \]
\[ \text{Alternative notations include \$C(n,k)\$,} \]
\[ \text{\$\{nC_k\}$, \$\{nC_k\}$, and \$C^n_k\$.} \]
The probability mass function for this random variable is

\[ P(X = k) = \begin{cases} \frac{1}{b-a}, & b \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \]

TABLE I
A table of common and important random variables.

<table>
<thead>
<tr>
<th>Name</th>
<th>Density Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>( f_X(x) = \lambda e^{-\lambda x} )</td>
</tr>
<tr>
<td>Laplace</td>
<td>( f_X(x) = \frac{1}{2} \alpha e^{-\alpha</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>( f_X(x) = \alpha^2 x e^{-\alpha^2 x^2/2}, \ x \geq 0 )</td>
</tr>
<tr>
<td>Uniform</td>
<td>( f_X(x) = 1/(b-a), \ b \leq x \leq a )</td>
</tr>
</tbody>
</table>

Two properties of the density function are:
1) \( f_X(x) \geq 0 \) for all \( x \).
2) \( \int_{-\infty}^{\infty} f_X(x) dx = 1 \)

Two properties of the density function are:
\begin{align*}
& f_X(x) \geq 0 \text{ for all } x. \\
& \int_{-\infty}^{\infty} f_X(x) dx = 1
\end{align*}
\begin{array*}
|M_X(j\omega)| &= \left| \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx \right| \\
&\leq \int_{-\infty}^{\infty} |e^{j\omega x} f_X(x)| dx \\
&= \int_{-\infty}^{\infty} |e^{j\omega x}| |f_X(x)| dx = \int_{-\infty}^{\infty} f_X(x) dx = 1
\end{array*}

then the characteristic function is well-defined and will always exist for any probability density function.

B. Gaussian Random Variable

A zero-mean Gaussian random variable $X$ has a density function of the form

$$f_X(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-x^2 / 2\sigma_x^2}$$

where $\sigma_x^2$ is the variance of $X$. A plot of the density function of a Gaussian for several different values of $\sigma_x$ is shown in Fig. 1.

\section{Gaussian Random Variable}

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where $\sigma_x^2$ is the variance of $X$. A plot of the density function of a Gaussian for several different values of $\sigma_x$ is shown in Fig. 1.
The characteristic function is

\[ MX(j\omega) = \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x} e^{-x^2/2\sigma_x^2} \]

\[ = e^{-\omega^2 \sigma_x^2/2} \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x - j\omega \sigma_x)^2/2\sigma_x^2} dx \]

\[ = 1 \]

The characteristic function is

\begin{align*}
M_X(j\omega) &= \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x} e^{-x^2/2\sigma_x^2} \\
&= e^{-\omega^2 \sigma_x^2/2} \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x - j\omega \sigma_x)^2/2\sigma_x^2} dx = 1
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\end{align*}
\]

**Double Integrals**

\[ P\{x_1 \leq X \leq x_2, \ y_1 \leq X \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x, y) dx dy \]

\[ \text{(6)} \]

\begin{align*}
P\{x_1 \leq X \leq x_2, \ y_1 \leq X \leq y_2\} &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x, y) dx dy \\
&= \text{IntegrateJointDensity}
\end{align*}
For $n$ random variables, the correlation matrix has the form
\[
R_X = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1n} \\
    r_{21} & r_{22} & \cdots & r_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{n1} & r_{n2} & \cdots & r_{nn}
\end{bmatrix}
\]
Unlike the joint distribution function, which is constrained by the requirement that the integral of the joint density function over $\mathbb{R}$ is equal to one, there is no such constraint on the joint density function. In fact, $f_{XY}$ may be arbitrarily large as long as the integral of $f_{XY}$ over all $x$ and $y$ is equal to one.

The classic work in the field is the text by Papoulis [1]. Another recommended text is [2]. An introduction to Monte Carlo simulations may be found in [3].

If we consider these two random variables to be a random vector, then the sum of their squares,

$$E[Z^2] = E[X^2] + E[Y^2],$$

is equal to the sum of the variances of the random variables and the variance of the sum of the random variables is equal to the correlation of the random variables.

The correlation matrix has the form

$$R = \begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix},$$

where $r_{xx} = E[(X - E[X])^2]$ and $r_{yy} = E[(Y - E[Y])^2]$.

These correlations are often visualized as correlations in the correlation matrix.

For two or more random variables, an important ensemble average is the correlation. Given two random variables $X$ and $Y$, the correlation is defined by

$$r_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}}.$$

The correlation matrix can be estimated from a set of realizations (experimental outcomes) of the random variables as

$$\hat{r}_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}}.$$

The sample autocorrelation at lag $k$ is defined by

$$\hat{r}_{xy}(k) = \frac{\sum_{i=1}^{n-k} (x_i - \bar{x})(y_{i+k} - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}}.$$

In the discrete case, the correlation is defined by

$$\hat{r}_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}}.$$

In the continuous case, the correlation is defined by

$$\hat{r}_{xy} = \frac{\int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f(x,y) \, dx \, dy}{\sqrt{\int_{-\infty}^{\infty} (x - \bar{x})^2 f(x,y) \, dx \, dy \int_{-\infty}^{\infty} (y - \bar{y})^2 f(x,y) \, dx \, dy}}.$$

Fig. 2. The Chi-square Random Variable.

\begin{figure}
\centering
\subfigure[Density Function]{\includegraphics[width=0.45\hsize]{images/Chi-Square_distributioncDF.png}} \hspace{1cm}
\subfigure[Distribution Function]{\includegraphics[width=0.45\hsize]{images/Chi-Square_distributionPDF.png}}
\caption{The Chi-square Random Variable.}
\label{fig:ChiSquare}
\end{figure}
VIII. CONCLUSION

There are many excellent textbooks where the reader may find advanced developments of the results presented in this paper. The classic work in the field is the text by Papoulis [1]. Another recommended text is [2]. An introduction to Monte Carlo simulations may be found in [3].

REFERENCES

Manual Specification of References

\begin{thebibliography}{9}
\bibitem{MonteCarlo} S. Raychaudhuri, "Introduction to Monte Carlo Simulation," \emph{Simulation Conference}, pp. 91-100, Dec. 2008
\end{thebibliography}

\section*{IdThen Package}

\%---------Ignore all text from here until \fi---------
\%---Replace \iffalse with \iftrue to include text -----
\iffalse
It is clear that this probability assignment satisfies the first probability axiom since all probabilities in Eq.~\ref{eq:prob_assgn} are positive.
\fi
\%----------------end -----------------

\section*{Verbatim Text}

\begin{verbatim}
\begin{align*}
M_X(j\omega ) &= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x} e^{-x^2/2\sigma_x^2} \underbrace{\frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-j\omega \sigma_x)^2/2\sigma_x^2} dx}_{=1}
\end{align*}
\end{verbatim}

To include computer listings or other similar text, we would like to have unformatted text to produce something like:
\begin{verbatim}
\begin{verbatim}
\begin{align*}
M_X(j\omega ) &= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x} e^{-x^2/2\sigma_x^2} \\
&= e^{-\omega^2 \sigma_x^2/2} \underbrace{\frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-j\omega \sigma_x)^2/2\sigma_x^2} dx}_{=1}
\end{align*}
\end{verbatim}
\end{verbatim}
\end{verbatim}

There are several ways to introduce text that won’t be interpreted by the compiler.

\begin{itemize}
\item With the \verbatim environment, everything input between a \verbatim and an \endverbatim command will be processed as if by a typewriter.
\item Also see the \verb command for short in-line verbatim text.
\end{itemize}
Graphics

Now that you have your beautifully typeset journal paper or article, you want your figures, block diagrams, plots and other graphics to be beautifully typset.

There are a number of very powerful packages that allow you to create graphics in postscript or PDF file format. Some of these are:

- xfig,
- TikZ and PGF,
- XY-Pic,
- PSTricks and PDFTricks,
- Metapost
- Adobe Illustrator

See the web for a description of these packages and for documentation.

Next Time

- How to prepare and deliver an effective presentation.
- Presentations with PowerPoint Using Aurora
- \LaTeX\ Presentations using the Beamer and Prosper Classes