All linear methods are equal—and extendible to (some) nonlinearities

Janos Gertler*†

School of Information Technology and Engineering, George Mason University, Fairfax, VA 22030, USA

SUMMARY
Several linear methods of residual generation for fault detection and diagnosis are reviewed. The parity relation approach is introduced in some detail, for both additive and parametric faults. The Chow–Willsky scheme, various diagnostic observers and principal component analysis are compared to the additive version. The ‘local approach’ and the least-squares estimation of parameter changes are shown to be related to the parametric variant. Nonlinear extensions are demonstrated for all the techniques under additive faults. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: fault detection and diagnosis; nonlinear systems; parity relations; diagnostic observers; principal component analysis; parameter estimation

1. INTRODUCTION
The past 15 years have witnessed significant activity in the field of model-based fault detection and diagnosis. Much of the work has been devoted to the generation of residuals, in a framework referred to as analytical redundancy. Residuals are quantities expressing the difference between the actual plant outputs and those expected on the basis of inputs and the mathematical plant model. These residuals carry the information allowing the detection of faults and, with appropriate manipulations, may also facilitate their isolation. For historical reference, the reader is directed to a succession of survey papers References [1–4].

While a multitude of residual generation methods has been proposed in the literature, two important tendencies may be observed: a certain convergence of the linear methods and their extension to nonlinear systems. The latter is a natural development, given that most real-life plants are nonlinear. The unification of methods, or at least the revealing of their interrelationships, is very significant, not only for our scientific curiosity, but also because

*Correspondence to: Janos Gertler, School of Information Technology and Engineering, George Mason University, Fairfax, VA 22030, U.S.A.
†E-mail: jgertler@gmu.edu

Contract/grant sponsor: NSF; contract/grant number: ECS-9906250

Copyright © 2002 John Wiley & Sons, Ltd.
techniques developed in the framework of one method may thus be transplanted into another framework; certain behaviours and predicted for another; and expectations and claims about the performance of a ‘new’ approach may be placed on a more realistic ground. In this paper, we will explore and review the links between the main classes of linear residual generation methods, and will also demonstrate their extension to nonlinear models. Our frame of reference will be the parity (consistency) relation method. We will develop this both for additive faults and for parametric faults. The other methods we will consider for additive faults include:

- the Chow–Willsky scheme,
- diagnostic observers, and
- principal component analysis.

For parametric faults, we will also investigate:

- the statistical ‘local approach’ and
- the least-squares estimation of parameter changes.

We will show that, in several cases, the listed methods lead to residuals which are identical with the parity relation residuals. This implies that there is no difference in the performance (sensitivity, robustness) of the methods. In some other instances, the parity relation residuals will be shown to be limiting cases of the residuals obtained from the other methods. We will also touch upon the issues of fault detectability and isolability, which are system properties and thus independent of the method of residual generation.

2. SYSTEM DESCRIPTION

We are dealing with systems depicted in Figure 1, where \( u(t) \) is the vector of known inputs (measured or manipulated), \( y(t) \) is the \( m \) vector of known outputs (measured), \( p(t) \) contains the additive faults and \( q(t) \) the additive disturbances and \( v(t) \) is a vector of noise. Also, \( \pi \) is the nominal value of physical parameters which may be subject to parametric faults \( \Delta \pi \).
**Linear systems:** A discrete linear input–output description of the system is

\[ y(t) = F(z, \pi + \Delta \pi)u(t) + S_p(z)p(t) + S_q(z)q(t) + S_v(z)v(t) \]  

(1)

where \( F(z), S_p(z), S_q(z) \) and \( S_v(z) \) are transfer functions in the shift operator \( z \). We ignore second-order effects, such as the dependence of \( S_p(t) \) and \( S_q(t) \) on the parametric faults \( \Delta \pi \).

A state-space representation of the linear system is

\[
\begin{align*}
    x(t + 1) &= Ax(t) + Bu(t) + B_p p(t) + B_q q(t) + B_v v(t) \\
    y(t) &= Cx(t) + Du(t) + D_p p(t) + D_q q(t) + D_v v(t)
\end{align*}
\]

(2)

where the matrices \( A, B, C \) and \( D \) are normally considered to depend on the parameters and the faults associated with them, while the dependence on the latter of \( B_p, B_q, B_v, D_p, D_q, D_v \) is usually ignored.

An important special case arises when we are concerned only with actuator and sensor faults, represented as additive faults \( \Delta u_A(t), \Delta u_M(t) \) and \( \Delta y(t) \), as shown in Figure 2. Then

\[
p(t) = \begin{bmatrix} \Delta y'(t) & -\Delta u_A'(t) & -u_M'(t) \end{bmatrix}^T
\]

(3)

\[ S_p(z) = [I - F(z, \pi)] \]

(4)

\[ B_p = [0 - B], \quad D_p = [I 0] \]

(5)

**Nonlinear systems:** If the system is nonlinear, its input–output description in general is in the implicit form

\[ f^S[y(t-), u(t-), p(t-), q(t-), v(t-), \pi + \Delta \pi] = 0 \]

(6)

where \( f^S[...] \) represents a set of nonlinear relations and the symbol \((t-)\) indicates that the variable is present, in addition to its current value, with some past samples as well. In some cases, (6) may be made explicit for the outputs as

\[ H(z) y(t) = f[u(t-), p(t-), q(t-), v(t-), \pi + \Delta \pi] \]

(7)

where \( f[...] \) is a set of nonlinear relations and \( H(z) \) is a polynomial matrix.

---

**Figure 2.**

Copyright © 2002 John Wiley & Sons, Ltd.

*Int. J. Robust Nonlinear Control* 2002; 12:000–000
A general nonlinear state-space model would have the form

\[ x(t+1) = \phi[x(t), u(t), p(t), q(t), v(t), \pi + \Delta \pi] \]
\[ y(t) = \phi[x(t), u(t), p(t), q(t), v(t), \pi + \Delta \pi] \] (8)

Such general models, however, are intractable. A relatively simple class of nonlinear systems arises when only the inputs (and outputs) in the state equation enter nonlinearly, via output injection

\[ x(t+1) = Ax(t) + \psi[y(t), u(t), p(t), q(t), v(t), \pi + \Delta \pi] \]
\[ y(t) = Cx(t) + Du(t) + D_p p(t) + D_q q(t) + D_v v(t) \] (9)

where the coefficient matrices \( A, C \) and \( D \) may also depend on the (uncertain) parameters \( \pi + \Delta \pi \). The input–output model corresponding to (9) is

\[ y(t) = C(zI - A)^{-1}\psi(t) + Du(t) + D_p p(t) + D_q q(t) + D_v v(t) \] (10)

where \( \psi(t) = \psi[y(t), u(t), p(t), q(t), v(t), \pi + \Delta \pi] \), which is a special case of the explicit nonlinear model. Other nonlinear state-space models will be briefly discussed in Section 6.

3. RESIDUAL PROPERTIES

Residuals are designed for the following requirements:

1. Disturbance decoupling: Residuals have to be insensitive to the disturbances while maintaining sensitivity with respect to (some) faults.
2. Isolation enhancement: Residuals need to have special properties to support fault isolation. Two isolation enhancement approaches are widely used:
   - Directional residuals: In response to a particular fault, the residual vector lies in a fault-specific direction, at all times.
   - Structured residuals: Each residual is sensitive to a subset of faults while insensitive to the rest. In the presence of a particular fault, a fault-specific subset of residuals will respond, yielding a unique fault code.
   - Diagonal residuals: whereas each residual responds to a single fault, are a special case of both the directional and the structural designs.
3. Resilience to noise: It should be possible to detect and isolate faults from the residuals even in the presence of (non-excessive) noise.

While a single residual may be sufficient to detect faults, isolation requires a set (vector) of residuals. Disturbance decoupling may be incorporated into the enhancement schemes; with directional residuals, disturbances are ‘faults’ with zero-vector responses, while with structured residuals, they are ones from which all residuals must be decoupled.

Structured residuals are usually characterized by a binary structure (incidence) matrix. The rows belong to residuals and the columns to faults. A 0 in an intersection means that the residual is decoupled from the fault, a 1 means it is not. The columns of the matrix are the fault codes. For isolation, all columns must be different; for ‘strong’ isolation, no column should turn into
another by replacing 1’s with 0’s. In a symmetrical strongly isolating structure, each column has the same number of 0’s, each in a different pattern.

4. PARITY RELATIONS

4.1. Additive faults in linear systems

The primary residuals from (1) are

\[
e(t) = y(t) - F(z, \pi)u(t) = S_p(z)p(t) + S_q(z)q(t) + S_e(t)v(t)
\]

Here the first line is the computational form of the residual, while the second line is its internal form, indicating the effect of faults, disturbances and noise. Enhanced residuals are obtained by the transformation

\[
r(t) = W(z)e(t)
\]

The transformation \( W(z) \) is so chosen that the residuals satisfy the response specification

\[
r(t) = N_p(z)p(t) + N_q(z)q(t)
\]

where \( N_p(z) \) is the desired fault response and \( N_q(z) = 0 \). This leads to the design equation

\[
W(z)[S_p(z) \quad S_q(z)] = [N_q(z) \quad 0]
\]

With \( m \) outputs, each row \( w_i(z) \) of \( W(z) \) has \( m \) elements, so it may be designed to satisfy up to \( m \) independent conditions. Thus (11) has a solution, for arbitrary \( N_p(z) \), only if

\[
\text{Rank } S_q(z) + \text{Rank } S_p^i(z) \leq m
\]

where \( S_p^i(z) \) is a submatrix of \( S_p(z) \) which belongs to a subset of faults. Note that in case of equality in (15) the solution is unique.

4.1.1. Directional design

Take a (sub)set of faults satisfying (15) and specify the responses as

\[
N_p^i(z) = \gamma(z)[\beta_1 \ldots \beta_k]
\]

where \( \beta_1 \ldots \beta_k \) are independent response directions. Then (14) may be solved for \( W(z) \). In the most important case when \([ S_p^i(z) \quad S_q(z) ] \) is square and full rank, the solution is

\[
W(z) = [N_p^i(z) \quad 0][S_p^i(z) \quad S_q(z)]^{-1}
\]

If there are additional faults, outside the subset with independent response directions, their response will also be directional but not independent.

4.1.2. Structured design

Since each scalar residual \( r_i(t) \) is decoupled from a different subset of faults, each row transformation \( w_i(z) \) has to be designed separately. While the maximum number of decouplings is desirable, at least one non-zero-fault response has to be specified to avoid complete decoupling. Choose a subset of faults so that \([ S_p^i(z) \quad S_q(z) ] \) satisfies (15), where \( S_p^i(z) \) belongs to faults from which \( r_i(t) \) is to be decoupled and \( s_{p,j}(z) \) to \( p_j \) for which nonzero response
is specified, and specify the responses as

\[ n_{pi}(z) = [y_j(z) \ 0 \ \ldots \ \ 0] \] (18)

Then, the single-row equivalent of (14) may be solved for \( w_i(z) \). In the most important case when \([s_{p,i}(z) \ S'_{p,i}(z) \ S_q(z)]\) is square and full rank, the solution is obtained as

\[ w_i(z) = [n_{pi}(z) \ 0'] [s_{p,i}(z) \ S'_{p,i}(z) \ S_q(z)]^{-1} \] (19)

The residual \( r_i(t) \) will not be decoupled from any fault \( p_{\theta} \) outside the decoupling subset, provided

\[ \text{Rank}[s_{p,i}(z) \ S'_{p,i}(z) \ S_q(z)] = \text{Rank}[S'_{p}(z) \ S_q(z)] + 1 \] (20)

If (20) is not satisfied, the selected row structure is not attainable but other structures may be chosen.

Detectability and isolability: It is clear from (14) that a fault \( p_j \) is not detectable if

\[ \text{Rank}[s_{p,j}(z) \ S_q(z)] = \text{Rank} S_q(z) \] (21)

because then any residual decoupled from the disturbances will be decoupled from \( p_j \) as well. This with (15) implies that no fault may be detected if \( \text{Rank} S_q(z) = m \). For additional criteria see Reference [5]. Further, faults \( p_j \) and \( p_{\theta} \) cannot be isolated from each other if

\[ \text{Rank}[s_{p,j}(z) \ s_{p,\theta}(z)] = 1 \] (22)

because then it is not possible to design directional residuals for them with separate response directions, nor any structured residual which is decoupled from one but not from the other.

Implementation: The implementation of the design equations (17) and (19) is a non-trivial task, for:

- in order to guarantee that the residual generator is causal and stable one may need to modify the response specification;
- the direct computation of the inverse transfer function involves algebraic pole-zero cancellations.

A systematic procedure based on the fault system matrix resolves these difficulties Reference [6]. Here, we will summarize the main results, in somewhat simplified form.

Define the fault system matrix

\[ \Gamma(z) = \begin{bmatrix} zI \ A & -B^*_p & -B_q \\ C & D^x_p & D_q \end{bmatrix} \] (23)

and compute its inverse

\[ \Gamma^{-1}(z) = \frac{\Omega(z)}{\zeta(z)} = \begin{bmatrix} \Omega_A(z) & \Omega_B(z) \\ \Omega_C(z) & \Omega_D(z) \end{bmatrix} \] (24)

where \( \zeta(z) \) is the invariant zero polynomial and \( \Omega(z) \) is the polynomial adjoint matrix. It can be shown that, for any row of (17) or (19),

\[ w_i(z) = [n'_{pi}(z) \ 0'] \Omega_D(z) / \zeta(z) \] (25)

\[ w_i(z)F(z) = [n'_{pi}(z) \ 0'] [\Omega_D(z)D - \Omega_C(z)B] / \zeta(z) \] (26)
Define \( \omega_j^f(z) \) as the \( j \)th row of \([\Omega_C(z) \ \Omega_D(z)]\). Then the following properties arise:

**Causality:** The response \( n_{pj}(z) \) must contain \( z^{-\delta} \) where \( \delta \) is the excess degree of \( \omega_j^f(z) \) over \( \zeta(z) \).

**Stability:** \( n_{pj}(z) \) must contain any unstable pole factors which appear in \( \zeta(z) \) but not in \( \omega_j^f(z) \).

**Detectability in steady state:** A fault \( p_j \) is undetectable in steady state if \( n_{pj}(z) \) contains the pole factor \( (z - 1) \), due to stabilization.

The order of the residual generator: is the degree of \( \zeta(z) \), which is \( v - \kappa \), where \( v \) is the system order (the size of \( A \)) and \( \kappa \) is the number of strictly input faults (the number of zero columns in \([D_p^e \ D_q^e] \)).

**Polynomial residual generators:** are ones in which both the fault responses and the transformations are polynomial. These can be obtained in the general design procedure, by including in \( n_{pj}(z) \) all factors of \( \zeta(z) \) which do not appear also in \( \omega_j^f(z) \). Alternatively, the design may rely on polynomial primary residuals. Write the transfer function as

\[
F(z) = G(z)/h(z)
\]

where \( G(z) \) is a polynomial matrix and \( h(z) \) is a polynomial. Then

\[
e^*(t) = h(z)y(t) - G(z)u(t)
\]

is the vector of polynomial primary residuals. Applying a polynomial transformation \( W^*(z) \) to these, further polynomial residuals

\[
r^*(t) = W^*(z)e^*(t)
\]

are obtained; with the appropriate choice of \( W^*(z) \), these are decoupled from the disturbances and enhanced for the isolation of faults.

**Similarity:** If a polynomial structured residual is designed so that its only specified non-zero response is to \( p_j \) then the transformation does not depend on the \( j \)th column of \( B_p^e \) and \( D_p^e \). Thus, for a given row structure, all transformations \( w_i(z) \) are similar, differing only in a polynomial factor determined by the response specification \( n_{pi}(z) \).

### 4.2. Additive faults in nonlinear systems

Recall the nonlinear system models (6) and (7) and ignore the noise \( v(t) \) and the parametric faults \( \Delta \pi \). The primary residuals are computed as exact nonlinear relations; for the explicit model (7) as

\[
e(t) = H(z)y(t) - f_0[u(t), p(t), q(t), \pi]
\]

and for the implicit model (6) as

\[
e(t) = f_0^i[u(t), y(t), p(t), q(t), \pi]
\]

where in \( f_0[u(t), p(t), q(t), \pi] \) and \( f_0^i[u(t), y(t), p(t), q(t), \pi] \), 0 values are substituted for the faults \( p(t) \), and disturbances \( q(t) \). For disturbance decoupling and isolation enhancement, we are reviewing several approaches.

**Fault-effect linearization:** The effects of the faults and disturbances in the residuals (30) and (31) can be expressed (approximately) with first derivatives. For the explicit form (30), utilizing that \( H(z)y(t) = f[\ldots] \),

\[
e(t) = f[\ldots] - f_0[\ldots] = M_p(t)p(t) + M_q(t)q(t)
\]
The derivative of the shifted function, relative to \( p \), is
\[
M_p(t) = \left. \frac{\partial f[\ldots]}{\partial p} \right|_{p=0}
\]
and relative to \( q \),
\[
M_q(t) = \left. \frac{\partial f[\ldots]}{\partial q} \right|_{q=0}
\]  \hspace{1cm} (33)
are Jacobian matrices. For the implicit form (31), utilizing that \( f^2[\ldots] = 0 \),
\[
e(t) = f^2_0[\ldots] - f^2_1[\ldots] = -M^\delta_p(t)p(t) - M^\delta_q(t)q(t)
\]  \hspace{1cm} (34)
where \( M^\delta_p(t) \) and \( M^\delta_q(t) \) are the Jacobians obtained for the implicit model. Equations (32) and (34), together with the computational forms (30) and (31), are in fact nonlinear parity relations. The derivative matrices \( M_p(t) \) and \( M_q(t) \) play the same role as \( S_p(z) \) and \( S_q(z) \) in the linear case; they can be used to perform disturbance decoupling and to design directional or structured residuals. Note, however, that \( M_p(t) \) and \( M_q(t) \) depend on the variables and are time-varying, therefore the transformations are also time-varying.

Equations (32) and (34) involve two approximations: Linearization of the disturbance and fault effects and time-averaging over the dynamic range of the model.

Note that for the special explicit model (10), arising from output injection, the computational form of the primary residual is
\[
e(t) = y(t) - C(zI - A)^{-1}\psi_q(t) - Du(t)
\]  \hspace{1cm} (35)
and its linearized fault-effect form is
\[
e(t) = \left[ C(zI - A)^{-1}\frac{\partial \psi(t)}{\partial p(t)} + D_p \right] p(t) + \left[ C(zI - A)^{-1}\frac{\partial \psi(t)}{\partial q(t)} + D_q \right] q(t)
\]  \hspace{1cm} (36)
The coefficients in this expression vary with time and thus the shift operator is being applied to bilinear expressions. This can be handled utilizing the following observations.

Consider a nonlinear function \( \psi(t) = \psi[y(t), u(t), p(t), q(t)] \) differentiable with respect to \( p(t) \) and \( q(t) \). The derivative of this, e.g. with respect to \( p(t) \) (scalar or vector) is \( \gamma(t) = \frac{\partial \psi(t)}{\partial p(t)} \). Now apply the shift operator to \( \psi(t) \), as
\[
z^{-\tau}\psi(t) = \psi(t - \tau) = \psi[y(t - \tau), u(t - \tau), p(t - \tau), q(t - \tau)]
\]  \hspace{1cm} (37)
The derivative of the shifted function, with respect to \( p(t - \tau) \), is \( \frac{\partial \psi(t - \tau)}{\partial p(t - \tau)} = \gamma(t - \tau) \). Then the derivative of the shifted function relative to \( p(t) \) is formally obtained as
\[
\frac{\partial \psi(t - \tau)}{\partial p(t)} = \frac{\partial \psi(t - \tau)}{\partial z^{-\tau} p(t)} \cdot z^{-\tau} = \frac{\partial \psi(t - \tau)}{\partial p(t - \tau)} \cdot z^{-\tau} = \gamma(t - \tau) \cdot z^{-\tau}
\]  \hspace{1cm} (38)
This is the same as saying
\[
z^{-\tau}\gamma(t) p(t) = z^{-\tau}[\gamma(t) p(t)] = \gamma(t - \tau) p(t - \tau) = \gamma(t - \tau) z^{-\tau} p(t)
\]  \hspace{1cm} (39)

**Algebraic elimination:** If the model is polynomial, as it is often the case, and the objective is to generate structured residuals, then nonlinear algebraic elimination is a possible alternative. This approach is most natural when we are concerned only with sensor and actuator faults because these get eliminated together with the variables they accompany. With the algebraic approach, the elimination of faults is perfect. But the procedure is rather complex (though symbolic computer packages are now available) and so is the resulting residual structure Reference [7].

**Direct identification:** Any structured residual can be thought of as arising from the model of a subsystem, which contains only a subset of the system variables. Another possible approach to generating such residuals in a nonlinear framework, especially for sensor and actuator faults, is
the direct identification of these subsystem models. Empirical data is obtained from the physical plant, or from the nonlinear input–output model used as a simulator. The ‘coarse structure’ of the submodel, that is, the variables present, is determined by the respective row of the structure matrix. The submodel is usually polynomial, its ‘fine structure’, that is the polynomial terms used, is determined empirically in the course of identification. This approach avoids the algebraic difficulties of the previous two methods and is particularly justified if the nonlinear input–output model is itself an empirical polynomial model. The direct identification approach has been applied successfully to automotive engines Reference [8].

5. THE CHOW–WILLSKY SCHEME

The celebrated Chow–Willsky scheme [9], which many people consider the parity relation approach, in fact addresses an important but special problem: the generation of polynomial structured residuals from the state-space model. It relies on the state-space model (2), with the noise-term and parameter dependence ignored; the state equation is applied recursively and substituted into the output equation

\[ y(t - \sigma) = Cx(t - \sigma) + Du(t - \sigma) + D_p p(t - \sigma) + D_p q(t - \sigma) \]

\[ y(t - \sigma + 1) = CAx(t - \sigma) + [CBu(t - \sigma) + Du(t - \sigma + 1)] + [CB_p p(t - \sigma)] \]

\[ + D_p p(t - \sigma + 1)] + [CB_q q(t - \sigma) + D_p q(t - \sigma + 1)] \]

\[ \ldots \]

\[ y(t) = CA^\sigma x(t - \sigma) + [CA^{\sigma-1} Bu(t - \sigma) + \cdots + CBu(t - 1) + Du(t)] \]

\[ + [CA^{\sigma-1} B_p p(t - \sigma) + \cdots + CB_p p(t - 1) + D_p p(t)] \]

\[ + [CA^{\sigma-1} B_q q(t - \sigma) + \cdots + CB_q q(t - 1) + D_p q(t)] \]

where \( \sigma \), the window width, is a design parameter. This set of equations is then put into the compact form

\[ Y(t) = Jx(t - \sigma) + LU(t) + L_p P(t) + L_q Q(t) \]

(41)

where

\[ Y(t) = [y'(t - \sigma) \quad y'(t - \sigma + 1) \quad \ldots \quad y'(t)]' \]

(42)

with \( U(t) \), \( P(t) \) and \( Q(t) \) defined similarly. Further,

\[ J = [(C) \quad (CA)' \quad \ldots \quad (CA^\sigma)']' \]

(43)

is the observability matrix, while \( L \), \( L_p \) and \( L_q \) are hyper-matrices composed of \( B \), \( C \), \( D \), \( B_p \), \( B_q \), \( D_p \) and \( D_q \) and powers of \( A \).

Primary residuals are obtained from (41) as

\[ E(t) = Y(t) - LU(t) = Jx(t - \sigma) + L_p P(t) + L_q Q(t) \]

(44)
These depend not only on the faults and disturbances but also on the state \(x(t - \sigma)\). Enhanced residuals are then computed as

\[
 r_i(t) = W_i[Y(t) - LU(t)]
\]

where \(W_i\) is a row vector. In order to eliminate the state from the residual, \(W_i\) must satisfy

\[
 W_i^tJ = 0
\]

while the remaining design freedom is used to decouple from the disturbances and some faults. A detailed analysis [10] shows that the maximum number of decouplings is \(m - 1\) and the minimum window-length to allow this is

\[
 \sigma = \nu - \kappa
\]

where \(\nu\) is the system order and \(\kappa\) is the number of zero columns in \([D_p^e \; D_q]\) (compare to order of generator in Section 4.1). Then the condition

\[
 W_i^t[J \quad L^0_p \quad L^0_q] = 0
\]

(where \(L^0_q\) is \(L_q\) without its 0 columns and \(L^0_p\) is the submatrix of \(L_p\) belonging to the decoupled faults, without its 0 columns) determines uniquely the direction (but not the size) of \(W_i\) (compare to similarity in Section 4.1).

We will rewrite now the Chow–Willsky generator in transfer function form. Decompose \(W_i\) as

\[
 W_i = [w_i, w_{i,\nu-1}, \ldots, w_i, 0]
\]

where the row vectors \(w_i\) are \(m\) long. Also

\[
 -W_i^tL = [v_i, v_{i,\nu-1}, \ldots, v_i, 0]
\]

Now define the polynomial vectors

\[
 w_i(z) = w_{i,\nu}z^\nu + w_{i,\nu-1}z^{\nu-1} + \cdots + w_i, 0
\]

\[
 v_i(z) = v_{i,\nu}z^\nu + v_{i,\nu-1}z^{\nu-1} + \cdots + v_i, 0
\]

Then

\[
 r_i(t) = w_i(z)y(t) + v_i(z)u(t)
\]

Further, if \(W_i\) was designed to satisfy (46) then, in the absence of faults and disturbances, \(r_i(t) = 0\). Thus \(v_i(z)\) must satisfy \(w_i(z)F(z)u(t) + v_i(z)u(t) = 0\). That is, \(v_i(z) = -w_i(z)F(z)\) and

\[
 r_i(t) = W_i(z)[Y(t) - F(z)u(t)]
\]

which is consistent with (11) and (12). Based on this, and on the generator order and similarity properties, we conclude that the scheme produces the same structured polynomial residual as the procedure described in Section 4.1. Further analysis of the scheme may be found in References [11,12,13].

**Nonlinear extension:** The Chow–Willsky scheme does not lend itself to easy nonlinear extension. However, it has been shown recently, for continuous systems, that such extension is possible if the nonlinearity is restricted to output injection Reference [14]. Here, we will demonstrate this idea using the discrete output injection model (9). The output-injection equivalent of Equation (41) is

\[
 Y(t) = Jx(t - \sigma) + K\Psi(t) + \Lambda U(t) + \Lambda_p P(t) + \Lambda_q Q(t)
\]
where
\[ \Psi(t) = [\psi'(t - \sigma) \quad \psi'(t - \sigma + 1) \ldots \quad \psi'(t)]' \]  

(56)

Further, \( K \) is a hyper-matrix formed of \( C \) and powers of \( A \), while \( \Lambda, \Lambda_p \) and \( \Lambda_q \) are block-diagonal matrices containing \( D, D_p \) and \( D_q \), respectively, as blocks. The primary residual is
\[
E(t) = Y(t) - K\Psi_0(t) - \Lambda U(t) = Jx(t - \sigma) + K[\Psi(t) - \Psi_0(t)] \\
+ \Lambda_p P(t) + \Lambda_q Q(t) 
\]

(57)

where \( \Psi_0(t) \) is \( \Psi(t) \) at \( p(t) = 0 \) and \( q(t) = 0 \). For structured design, the fault-effect form needs to be expanded with respect to the faults and disturbances, i.e.,
\[
E(t) = Jx(t - \sigma) + \left[ K \frac{\partial \Psi(t)}{\partial P(t)} + \Lambda_p \right] P(t) + \left[ K \frac{\partial \Psi(t)}{\partial Q(t)} + \Lambda_q \right] Q(t) 
\]

(58)

The partial derivatives are taken at \( P(t) = 0 \) and \( Q(t) = 0 \). The derivative matrices are very sparse but they depend on the operating point. Therefore, the transformation must be time-varying. The transformed residual is
\[
r_i(t) = W_i(t)E(t) = W_i(t) \left[ K \frac{\partial \Psi(t)}{\partial P(t)} + \Lambda_p \right] P(t) \\
+ W_i(t) \left[ K \frac{\partial \Psi(t)}{\partial Q(t)} + \Lambda_q \right] Q(t) 
\]

(59)

Here, we took into account that \( W_i(t)J = 0 \) for all \( t \).

6. DIAGNOSTIC OBSERVERS

Observer-based residual generators for additive faults have received significant attention in the diagnostic literature. They may be designed in the Luenberger framework or as unknown input observers. Structured residuals may be obtained in either case while directional design has been reported with the Luenberger observer. The placement of observer poles is part of the design.

6.1. Luenberger observer

The Luenberger observer is formulated as
\[
\dot{x}(t + 1) = Ax(t) + Bu(t) + Ko(t) \\
o(t) = y(t) - C\dot{x}(t) - Du(t) 
\]

(60)

The innovations \( o(t) \) may serve as primary residuals, with additional residuals computed as
\[
r(t) = Ho(t) 
\]

(61)

The input–output relationship obtained from (60) is
\[
o(t) = [I - T(z)K][y(t) - F(z)u(t)] 
\]

(62)

where
\[
T(z) = C(zI - A + KC)^{-1} 
\]

(63)
This can be considered as the computational form of a parity relation. With \( y(t) - F(z)u(t) \) recognized as \( e(t) \) and (61) taken into account, the equivalent parity relation transformation is

\[
W(z) = H[I - T(z)K]
\]

The fault-effect form, from (60) and (61) and with the disturbances included in \( p(t) \), is

\[
r(t) = H[T(z)(B_p - KD_p) + D_p]p(t)
\]

Thus, any Luenberger residual generator may be implemented as a set of parity relations. Further, if the design is based on specifications which lead to unique fault responses, then parity relations designed directly for those specifications yield the same residuals as the observer Reference [4].

**Structured design:** Structured residuals can be designed by eigenstructure assignment Reference [11]. Consider the state estimation error \( x(t) \) and ignore the disturbances. Assuming also that \( D_p = 0 \), this can be shown to be

\[
\dot{\zeta}(t) = x(t) - \dot{x}(t) = (zI - G)^{-1}B_p p(t)
\]

where \( G = A - KC \). Expand \( (zI - G)^{-1} \) into a series and decompose \( p \) into \( p' \) and \( p^{(0)} \), where the former contains the faults from which the residual \( r_i \) needs to be decoupled. Then (66) becomes

\[
\dot{\zeta}(t) = (z^{-1}I + z^{-2}G + z^{-3}G^2 + \cdots)[B_p p'(t) + B_p^{(0)} p^{(0)}(t)]
\]

Now choose \( K = K' \) so that the columns of \( B_p' \) are right-side eigenvectors of \( G = G' \), with zero eigenvalues. Then \( G'B_p' = 0 \) and the state-estimation error becomes

\[
\dot{\zeta}(t) = z^{-1}B_p^{(0)} p'(t) + (zI - G')^{-1}B_p^{(0)} p^{(0)}(t)
\]

The remaining freedom in choosing \( K \) is used to place the observer poles. Finally, the \( i \)th structured residual \( r_i(t) \) is computed as

\[
r_i(t) = h_i \sigma'(t) = h_i C \xi'(t)
\]

where \( h_i \) is chosen such that

\[
h_i CB_p' = 0
\]

thus completing the decoupling. If disturbances are also present, they are treated as elements of \( p_i \) for the entire residual vector.

**Directional design:** A Luenberger observer generating directional residuals is called the ‘detection filter’. It has been developed in the geometric framework Reference [15] and as eigenstructure assignment Reference [16]. Both techniques are far too complex to summarize here. Equivalent parity relations, in accordance with (61), have been demonstrated also for this filter Reference [17].

### 6.2. Unknown input observer

The unknown input observer is a generalization of the Luenberger algorithm, described as

\[
T \dot{x}(t + 1) = MT \dot{x}(t) + Ju(t) + Ny(t)
\]

\[
r(t) = L_1 T \dot{x}(t) + L_2 y(t)
\]
Reference [3]. With $D = 0$ and the disturbance included in $p(t)$, this leads to the error equations

$$T_x(t + 1) = MT_x(t) + (TA - MT - NC)x(t) + (TB - J)u(t) + TB_p p(t)$$

$$r(t) = -L_1 T_x(t) + (L_1 T + L_2 C)x(t)$$

(72)

For this algorithm to function as a residual generator, the following conditions must be satisfied:

$$TA - MT - NC = 0$$

$$TB - J = 0, \quad L_1 T + L_2 C = 0$$

(73)

By decomposing the fault vector $p(t)$ as in (67), decoupling from the subset $p'$ further requires

$$TB_p' = 0$$

(74)

The placement of poles is also part of the design. Solution techniques have been presented by References [3,18], and others. The input–output formulation of (71),

$$r(t) = L_1 (z I - M)^{-1} Ju(t) + [L_1 (z I - M)^{-1} N + L_2] y(t)$$

(75)

may be considered as an equivalent parity relation.

6.3. Nonlinear observers

For a detailed discussion of the subject, see Reference [19]; here, we only review a few basic ideas. A broad class of nonlinear systems can be characterized in state-space form as

$$x(t + 1) = A_0 x(t) + \phi[u(t), x(t)] + \psi[u(t), y(t)] + \varphi_p[u(t), p(t)]$$

$$y(t) = Cx(t) + Du(t) + D_p p(t)$$

(76)

where the disturbances have been included in the fault vector. The possibility to separate the fault effect from the other terms is a simplifying assumption. Even so, this formulation may be intractable, but a special case, when the $p$ functions are bilinear, has received significant attention. Now,

$$\phi[u(t), x(t)] = \sum_j u_j(t) A_j x(t) \quad \varphi_p[u(t), p(t)] = \sum_j u_j(t) B_{pj} p(t)$$

(77)

and the state equation in (76) can be rewritten as

$$x(t + 1) = A(u) x(t) + \psi[u(t), y(t)] + B_p(u) p(t)$$

(78)

For such a bilinear system, a Luenberger-type observer may be formulated as

$$\hat{x}(t + 1) = A(u) \hat{x}(t) + \psi[u(t), y(t)] + K(t) o(t)$$

$$o(t) = y(t) - C \hat{x}(t) - Du(t)$$

(79)

The design of such observers is far from trivial; results for two variants of (79) have recently been published by Reference [20].

The output injection model (9), which we repeat here, in simplified form, for easier reference

$$x(t + 1) = A x(t) + \psi[y(t), u(t), p(t)]$$

$$y(t) = C x(t) + Du(t) + D_p p(t)$$

(80)
is more restrictive than (76). For this system, the Luenberger observer is also simpler

\[
\dot{x}(t+1) = Ax(t) + \psi'[u(t), y(t), 0] + Ko(t)
\]

\[
o(t) = y(t) - CX(t) - Du(t)
\]

where 0 is substituted for the unknown fault vector \( p(t) \). From (81), an input–output relationship may be formally derived as

\[
o(t) = [I - T(z)K]\{y(t) - C(zI - A)^{-1}\psi'[u(t), y(t), 0] - Du(t)\}
\]

where \( T(z) \) is as defined in (63). Further, the \( \{,\ldots,\} \) expression in (82) can be recognized as the parity relation residual for the output injection system, compare to (35) and recall that \( \psi_0 \times (t) = \psi[u(t), y(t), 0] \). That is, the Luenberger observer designed for the output injection system can be related to an equivalent parity relation, just like in the linear case.

### 7. PRINCIPAL COMPONENT ANALYSIS

Principal component analysis (PCA) is a favoured tool in the process industries. By revealing linear dependences among variables, PCA may significantly reduce the dimensionality of the model for large-scale systems Reference [21]. Here we will describe the basic approach and will show how naturally PCA-based fault detection and isolation of additive (actuator and sensor) faults is linked to parity relations Reference [22]. We will start with static linear systems and then point out the extensions to dynamics and nonlinearities.

Consider a system with inputs and outputs \( u(t) = [u_1(t), \ldots, u_i(t)]' \) and \( y(t) = [y_1(t), \ldots, y_m \times (t)]' \) and assume we are concerned with sensor and actuator faults related to those variables, \( \Delta u(t) \) and \( \Delta y(t) \). The variables are linked by the model

\[
y(t) = Ax(t) + \Delta u(t) + \Delta y(t)
\]

Define \( x(t) = [y'(t), u'(t)]' \) (centred and normalized) and \( \Delta x(t) = [\Delta y'(t), \Delta u'(t)]' \) (normalized). Then

\[
Bx(t) = B\Delta x(t) \quad \text{where} \quad B = [I - A]
\]

To build a PCA model, collect fault-free data \( x^0(t), t = 1 \ldots N \). Construct the covariance matrix

\[
\Phi = \frac{1}{N} \sum_{t=1}^{N} x^0(t)x^0(t)
\]

Obtain the eigenvalues \( \sigma^2_1, \ldots, \sigma^2_{k+m} \) and eigenvectors \( q_1, \ldots, q_{k+m} \). Due to the \( m \) linear relations (and assuming the inputs are independent), \( m \) of the eigenvalues will be zero (or near zero, if there is noise and/or modeling errors). Thus, any fault-free observation can be expressed as

\[
x^0(t) = \hat{Q}s^0(t)
\]

where \( \hat{Q} = [q_1, \ldots, q_{m}] \) contains the eigenvectors with non-zero eigenvalues and \( s^0(t) = \hat{Q}x^0(t) \) is the projection of \( x^0(t) \) on those vectors.

When applied to \( x(t) \) and faults are present, (86) returns a non-zero residual

\[
x(t) - \hat{Q}s(t) = x(t) - \hat{Q}\hat{Q}x(t) = e(t) \geq 0
\]
Due to the orthogonality of eigenvectors, this may be written as
\[ e(t) = \mathbf{Q}' \Delta \mathbf{x}(t) \] (88)
where \( \mathbf{Q} = [\mathbf{q}_{k+1} \ldots \mathbf{q}_{k+m}] \) contains the eigenvectors with zero eigenvalues. \( e(t) \) is shown in the full space of \( \mathbf{x} \); expressed in the subspace spanned by \( \mathbf{q}_k \ldots \mathbf{q}_{k+m} \) it is
\[ e(t) = \mathbf{Q}' \mathbf{x}(t) \] (89)
With \( \mathbf{x}(t) = \mathbf{x}^0(t) + \Delta \mathbf{x}(t) \) and (86),
\[ e(t) = \mathbf{Q}' \mathbf{Q}^0\mathbf{s}^0(t) + \mathbf{Q}' \Delta \mathbf{x}(t) = \mathbf{Q}' \Delta \mathbf{x}(t) \] (90)
since \( \mathbf{Q}' \) and \( \mathbf{Q} \) are orthogonal.

Enhanced residuals: \( e(t) \) is the PCA primary residual, with (89) as its computational form and (90) as its fault-effect form. The latter is better seen as
\[ e(t) = [\mathbf{q}_{*1} \ldots \mathbf{q}_{*k+m}] [\Delta \mathbf{x}_1 \ldots \Delta \mathbf{x}_{k+m}]' \] (91)
where \( \mathbf{q}_{*j} \), \( j = 1 \ldots k + m \), are the columns of \( \mathbf{Q}' \). Additional residuals are then obtained by a transformation
\[ r(t) = \mathbf{W}e(t) \] (92)
Directional design is possible for any subset of \( m \) faults, as
\[ \mathbf{WQ}_x = [\beta_1 \ldots \beta_m] \] (93)
where \( \mathbf{Q}_x \) contains the columns \( \mathbf{q}_{*j} \) for the selected variables and \( \beta_j \) are the desired response directions. Structured design is possible for the full set, so that each scalar residual is decoupled from \( m - 1 \) faults, according to a structure matrix, with one non-zero response specified. Then the \( i \)th row transformation is
\[ w_i\mathbf{Q}' = [c_{ii} \ldots 0] \] (94)
where \( \mathbf{Q}' \) contains the columns for the variables with specified response in \( r_i(t) \).

Nonlinear structured PCA: An alternative to the algebraic transformation described above is the partial PCA approach. This technique follows the idea of direct identification of structured parity relations; subsets of variables are selected according to rows of a structure matrix and separate PCA models are established for each subset. These submodels are then selectively sensitive to subsets of faults, resulting in structured isolation Reference [23].

Partial PCA offers an easy extension to polynomial nonlinear models. The set of variables is expanded to include quadratic, bilinear, etc. combinations of the physical variables as pseudo-variables. When a variable is omitted from a particular subset, then all the pseudo-variables containing it go as well. Each partial model is built up gradually, starting with the linear terms and adding one nonlinear term at a time, until the desired accuracy is attained Reference [23].

Dynamic PCA: Dynamic PCA models are generated by including past observations of the physical variables as pseudo-variables, according to the dynamic order of the system. For structured residuals, partial PCA models are advantageous. The dynamic order is different for the various partial models and may be determined empirically, by the gradual buildup of the model. Note that over-parametrization (potential pole-zero cancellations) leads to extra linear relations in the partial model, which may interfere with structured isolation.
8. PARAMETRIC FAULTS IN PARITY RELATIONS

The input–output model usually depends on the physical parameters in a nonlinear way, whether the system is linear or nonlinear. Therefore, we will first handle parametric faults in the more general nonlinear situation.

Primary residuals: Recall the primary residuals (30) and (31). When these arise from parametric faults $\Delta \pi$ then the effect of these faults can be described as

$$e(t) = M_\pi(t)\Delta \pi$$  \hspace{1cm} (95)

in the explicit model (30) and

$$e(t) = -M_\pi^*(t)\Delta \pi$$  \hspace{1cm} (96)

in the implicit model (31). $M_\pi(t)$ is the Jacobian

$$M_\pi(t) = \frac{\partial f[\ldots]}{\partial \pi} \bigg|_{\Delta \pi = 0}$$  \hspace{1cm} (97)

with $M_\pi^*(t)$ obtained similarly from $f^*$. (95) and (96) clearly represent the linearization of the fault effect in the residual. The Jacobian depends on the variables and, via these, on time.

Linear systems: can be handled as special cases of (95) and (96) References [24,10]. Write the Jacobian for $k$ parametric faults with its columns as

$$M_\pi(t) = \begin{bmatrix} m_{\pi,1}(t) & \ldots & m_{\pi,k}(t) \end{bmatrix}$$  \hspace{1cm} (98)

Consider first the rational linear residual $e(t)$ in (11); the Jacobian for the $j$th parametric fault is

$$m_{\pi,j}(t) = \left[ \frac{\partial F(z, \pi)}{\partial \pi_j} \right] u(t)$$  \hspace{1cm} (99)

Notice that the derivatives of the rational transfer function matrix $F(z, \pi)$ are also rational matrices; these multiplied with the input vector $u(t)$ yield time-function vectors. Similarly, for the polynomial linear residual $e^*(t)$ in (28) the Jacobian is

$$m_{\pi,j}^* = \frac{\partial G(z, \pi)}{\partial \pi_j} u(t) - \frac{\partial h(z, \pi)}{\partial \pi_j} y(t)$$  \hspace{1cm} (100)

An important special case is when the primary residual set is obtained by the repeated shifting of a single polynomial residual

$$e^*(t) = Ze^*(t) = Zh(z) - Zg_1(z)u(t)$$

where

$$Z = \begin{bmatrix} 1 & z^{-1} & \ldots & z^{-K+1} \end{bmatrix}$$  \hspace{1cm} (101)

Enhanced residuals: for parametric faults are designed fundamentally the same way as for additive faults in linear systems (and exactly the same way as for additive faults in nonlinear systems), the Jacobian matrix $M_\pi(t)$ taking the place of the transfer function matrix $S_p(z)$. Notice, though, that $M_\pi(t)$ is a time-varying numerical matrix, which implies that the transformation needs to be done on-line but it is free from the issues of causality and stability.

Of particular interest is the case when $M_\pi(t)$ (or $M_\pi^*(t)$) is square. This may be achieved by making $m = k$ in a multiple-output situation or $K = k$ with a single output. Then residuals may be obtained for a specified $e(t) = N_\pi \Delta \pi$ response as

$$r(t) = M_\pi^{-1}(t)N_\pi e(t)$$  \hspace{1cm} (102)
In the special case when \( N_p = I \), the response is diagonal, \( e(t) = \Delta \pi \). An interpretation of this residual in relation to the least-squares estimation of the parameter changes will be given in Section 10.

Detectability and isolability: of parametric faults are linked to the rank properties of the Jacobian \( M_p(t) \). This is a time-varying matrix and its rank defects, if any, may be either permanent or temporary. They are determined primarily by the properties of the model but, with temporary rank-defects, the properties of the input data also play a role Reference [10].

9. LOCAL APPROACH TO PARAMETRIC FAULTS

The statistical local approach is a diagnostic philosophy due to Basseville and coworkers Reference [25]. Its essence is two fold:

(a) rather than estimating the size of a fault, its presence is detected (and isolated), using the assumption that the size is small;
(b) the problem is reduced to testing the hypothesis of zero versus nonzero mean of a Gaussian random vector.

Idea (a) is consistent with the way how faults are diagnosed in the parity relation framework. However, (b) requires residuals which have zero mean when there is no fault. The prediction error residual \( e(t) \) is not guaranteed to have this property in all situations. There are several other residuals which are, of which the gradient of the least-squares performance index

\[
J(t) = \frac{1}{2} \sum_{\tau=0}^{K-1} e'(t - \tau)e(t - \tau)
\]

is the most plausible choice. Now the primary residuals are defined as

\[
g_j(t) = \frac{\partial e(t)e(t)}{\partial \pi_j} = m_{\pi,j}(t)e(t), \quad j = 1 \ldots k
\]

where \( m_{\pi,j}(t) = [\partial F(z, \pi)/\partial \pi_j]u(t) \), see (99). The full primary residual vector is obtained as

\[
g(t) = M'_p(t)e(t)
\]

(c.f. (98)). For fault analysis, a filtered residual

\[
\zeta_N(t) = \frac{1}{\sqrt{K}} \sum_{\tau=0}^{K-1} g(t - \tau)
\]

is computed which is shown to have asymptotically Gaussian behaviour. Note that \( \zeta_N(t) \) is not an isolation enhanced residual. Fault detection and isolation are achieved by performing generalized likelihood ratio tests on this residual, the latter by assuming one fault (or a group of faults) at a time.

It is possible to design enhanced residuals by applying a transformation to \( g(t) \), but the zero mean guarantee may be lost. Obtain a residual as

\[
r(t) = W(t)g(t)
\]
and recall that \( e(t) \approx M_\pi(t)\Delta\pi \) (c.f. (95)). Then
\[
 r(t) = W(t)M_\pi'(t)M_\pi(t)\Delta\pi
\]  
(108)

For a specified response \( r(t) = N_\pi(t)\Delta\pi \), the transformation is \( W(t) = N_\pi(t)[M_\pi'(t)M_\pi(t)]^{-1} \) and
\[
r(t) = N_\pi(t)[M_\pi'(t)M_\pi(t)]^{-1}M_\pi'(t)e(t) = N_\pi(t)M_\pi^{-1}(t)e(t)
\]  
(109)

provided \( M_\pi(t) \) is square. This is the same as the standard parity relation for parametric faults, obtained in (102).

10. ESTIMATION OF PARAMETER CHANGES

System parameter estimation is a natural approach to the detection and isolation of parametric faults Reference [2]. Here, we will describe a method which estimates the changes in the physical parameters and will show how it is related to the parity relation approach References [25–28].

Consider the parameters \( \theta \) of the system model (transfer function) which depend on the physical parameters \( \pi \). Denote the actual value of the model parameters by \( \theta^0 \), in contrast to their nominal value \( \theta \). If there is a fault \( \Delta\theta \) then
\[
\theta^0 = \theta + \Delta\theta
\]  
(110)

In systems identification, a multiple-input–single-output relation is written in regression form as
\[
y_i(t) = \Phi_i(t)\theta^0
\]  
(111)

where \( \Phi_i(t) \) is the \( i \)th regression vector and \( \theta^0 \) is the vector of the actual values of the \( i \)th subsystem model. For multiple outputs, (111) becomes
\[
y(t) = \Phi(t)\theta^0
\]  
(112)

where \( \Phi(t) \) is a (sparse) regression matrix and \( \theta^0 \) contains the actual values of the model parameters for the entire system. If there is a fault then (112) applied to the nominal parameter values returns a nonzero residual (prediction error)
\[
e^\pi(t) = y(t) - \Phi(t)\theta = \Phi(t)\Delta\theta
\]  
(113)

which is the same as \( e^\pi(t) \) in (28).

The least-squares estimate of the parameters \( \theta^0 \), using a moving window of \( K \) observations (rather than exponential forgetting, for better response to sudden faults), is obtained as
\[
\hat{\theta} = \begin{bmatrix} \Psi(t) & \Psi(t) \end{bmatrix}^{-1}\Psi(t)Y(t)
\]  
(114)

where \( \Psi(t) \) and \( Y(t) \) contain stacked samples of \( \Phi(t) \) and \( y(t) \), for \( \tau = t, t-1, \ldots, t-K+1 \). Now from (113), \( Y(t) = \Phi^\pi(t) + \Psi(t) \), where \( \Phi^\pi(t) \) contains stacked samples of \( e^\pi(t) \). With this, (114) implies
\[
\Delta\hat{\theta} = \begin{bmatrix} \Psi(t) \end{bmatrix}^{-1}\Psi(t)\Phi^\pi(t)
\]  
(115)

That is, the parameter change is obtained directly from the parity relation residuals. (115) may also be considered as a ‘transformation’ between the residuals \( e^\pi(t) \) and \( \Delta\hat{\theta} \) Reference [26].

Behind changes of the model parameters \( \theta \), there are changes of the physical parameters \( \pi \). If the changes are small, they can be approximately related as
\[
\Delta\theta = R\Delta\pi, \quad -R = \dot{\pi}\hat{\pi}
\]  
(116)
With this, (113) becomes

$$e^s(t) = \Phi'(t)\Delta\theta = \Phi'(t)R\Delta\pi$$  \hspace{1cm} (117)

By analogy, replacing $\Psi(t)$ with $\Psi(t)R$ in (115) yields

$$\Delta\hat{\pi} = [R^\prime \Psi'(t)\Psi(t) R^{-1} R^\prime \Psi'(t) E^s(t)]$$  \hspace{1cm} (118)

which provides a direct estimation of the physical parameter changes. Note that $\pi$ usually contains much fewer elements than $\theta$, thus the persistent excitation requirements (rank conditions for $\Psi(t)$) are milder in (118) than in (115).

All the above results apply also to the single-output case (111). Now (118) becomes

$$\Delta\hat{\pi} = [R^\prime \Psi'_i(t)\Psi_i(t) R^{-1} R^\prime \Psi'_i(t) E^s_i(t)]$$  \hspace{1cm} (119)

with $R_i = \partial \theta_i / \partial \pi$ and $\Psi'_i(t)$ and $E^s_i(t)$ containing stacked samples of $\phi'_i(\tau)$ and $e_i(\tau) = y_i(\tau) - \phi'_i(\tau)\theta_i$.

Another link to parity relations: Applying (102) to $e^s(t)$ with $N_p = 1$ yields the residual $r \times (t) = [M^*_\pi(t)]^{-1} e^s(t)$, with diagonal fault response. But from (117),

$$M^*_\pi(t) = \partial e^s(t) / \partial \pi = \Phi'(t)R$$  \hspace{1cm} (120)

Thus the diagonal residual is

$$r(t) = [\Phi'(t)R]^{-1} e^s(t)$$  \hspace{1cm} (121)

Comparing this to (118) reveals that, with $\Psi(t) = \Phi'(t)$ and $k = m$ (for $m$ outputs and $k$ physical parameters), $r(t)$ is the ‘least-squares’ estimate of $\Delta\pi$ from a single observation of $\Phi'(t)$ and $e^s(t)$. Further, in the case of a single output,

$$r(t) = [\Psi'_i(t) R_i]^{-1} E^s_i(t)$$  \hspace{1cm} (122)

This can be recognized as the least-squares estimate from $K = k$ samples, which is the minimum window size allowing the inversion of $R_i^\prime \Psi'_i(t) \Psi_i(t) R_i$. Of course, such estimates are very noise sensitive.

11. CONCLUSIONS

The parity relation method has been reviewed, for both additive and parametric faults. It has been shown how the additive version is related to the Chow–Willsky scheme, diagnostic observers and principal component analysis. Links of the parametric version to the statistical local approach and to the least-squares estimation of parameter changes have been demonstrated. Several nonlinear extensions have also been pointed out.

ACKNOWLEDGEMENTS

This work was supported partially by NSF under Grant #ECS-9906250.

REFERENCES


Copyright © 2002 John Wiley & Sons, Ltd. *Int. J. Robust Nonlinear Control* 2002; **12**:000–000

Copyright © 2002 John Wiley & Sons, Ltd. *Int. J. Robust Nonlinear Control* 2002; 12:000–000