ECE 754  
JANUARY 21, 2009

* PLEASE TAKE 3 HANDOUTS

* FILL OUT QUESTIONNAIRE
Two Problems of Interest

Set of Distributed Sensors

Spatial Filtering/Beamforming

Spatial Spectrum Estimation

$y(t, \theta)$

$P(\omega, \theta)$

Application Areas

- Radar
- Sonar
- Astronomy
- Seismology
- Communications

Power as Function of Angle

Ultrasound

Hearing Aids
TOPICAL COVERAGE

* Basics, Intro to Spatial Filtering
* Deterministic Spatial Filtering
* Space-time Random Processes
  Statistical Model
* Optimum Beamforming
  Known Statistics $\rightarrow$ Performance Limits
* Adaptive Beamforming
  Estimate Statistics from Data
COORDINATE SYSTEM

θ = POLAR ANGLE

ϕ = AZIMUTHAL ANGLE

$r_i$ = RADIAL DISTANCE TO SENSOR

ϕ₁ = POSITION VECTOR

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} r_i \sin \theta \cos \phi \\ r_i \sin \theta \sin \phi \\ r_i \cos \theta \end{bmatrix}$$
If we were doing time-domain processing, what set of basis functions would you use?

Complex exponentials: \( e^{j\omega t} \)

- \( e^{j\omega t} \) is eigenfunction of LTI system

\[
 e^{j\omega t} \xrightarrow{LTI} e^{j\omega t} H(\omega_0)
\]

- Solutions to linear constant coefficient differential equation

- \( e^{j\omega t} \) foundation of Fourier analysis
Basis Functions $\rightarrow$ Plane Waves

Motivation: Plane Waves as solutions to wave eqn.

In free space, the scalar wave eqn.
is:

$$\nabla^2 f = \frac{1}{c^2} \frac{d^2 f}{dt^2}$$

$$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2} = \frac{1}{c^2} \frac{d^2 f}{dt^2}$$

$\sqrt{\text{Acoustic}}$

$\sqrt{\text{Electromagnetic}}$

$c =$ propagation speed in medium assumed constant

$\uparrow$ Partial differential eqn.
ASSUME A SEPARABLE SOLUTION:

\[ f(x, y, z, t) = F_{xW}(x) \cdot F_{yW}(y) \cdot F_{zW}(z) \cdot F_{tW}(t) \]

FOR SIMPLICITY, ASSUME

\[ f(x, y, z, t) = A e^{j\omega t - jk_x x - jk_y y - jk_z z} \]

\[ k_x, k_y, k_z, \omega \text{ ARE CONSTANTS, } \omega \geq 0 \]

\[ \frac{d^2 \{ \cdot \}}{dx^2} = -k_x^2 A e^{j\omega t - jk_x x - jk_y y - jk_z z} \]

SPATIAL \hspace{1cm} \uparrow \hspace{1cm} TEMPORAL
FREQUENCIES \hspace{1cm} FREQ.
\[-k_x^2 f(x, y, z, t) - k_y^2 f(x, y, z, t) - k_z^2 f(x, y, z, t) = \frac{-\omega^2}{c^2} f(x, y, z, t)\]

\[k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\]

When this constraint is satisfied, complex exponential signal satisfies wave Eqn.

\[j(\omega t - k_x x - k_y y - k_z z) \quad j(\omega t - k^T \varphi)\]

\[e^x \quad \text{and} \quad e^y \quad \text{and} \quad e^z\]

\[k = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \varphi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\]

Wave number vector

Position vector
$c (\omega t - k^T \phi)$

MONOCHROMATIC PLANEWAVE

MONOCHROMATIC = REFERS TO TIME BEHAVIOR OF THE SIGNAL. IT'S TIME SIGNAL WITH ONLY 1 FREQUENCY.

PLANEWAVE: $e^{-j k^T \phi}$

$k$ DEFINES DIRECTION OF PROPAGATION

WAVEFRONTS, LINES OF CONSTANT PHASE, PERPENDICULAR TO $k$. 
Constraint:  \[ |k| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\omega}{c} \]

(satisfies wave eqn.)

\[ |k| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \]

\( \lambda = \text{wave length} \)

\( \text{distance travelled in 1 temporal period} \)

\( \text{like spatial period} \)

Analogy:  time domain

\[ \Omega = \frac{2\pi}{T} \]
CAN WRITE $k$ AS FOLLOWS

$$k = \frac{2\pi}{\lambda} (-u) = -\frac{\omega}{c} u$$

$u =$ UNIT LENGTH = $\begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}$

$\rightarrow$ INTO ORIGIN

$\Rightarrow$ IN DIRECTION SPECIFIED BY $\theta, \phi$

$K =$ SPATIAL FREQ. ; 3-D SPACE

REQUIRES 3-ELEMENT VECTOR

$\Rightarrow$ WE'VE SHOWN THAT MONOCHROMATIC PLANEWAVES SATISFY WAVE EQN.
If we wanted to represent signal with more complicated temporal structure → use Fourier

Want pulse: \( f(t) = \int F(\omega) e^{j \omega t} d\omega \) Fourier synthesis

\[
[f(t) = \text{periodic, use Fourier series}...]
\]

Do same thing in spatial domain:

\[
f(t, \phi) = \iiint F(\omega, k) e^{j (\omega t - k^T \phi)} \, dw \, dk \, d\omega \, d\phi
\]

\[
F(\omega, k) = \iiint f(t, \phi) e^{-j (\omega t - k^T \phi)} \, dt \, d\phi
\]
SUMMARY: WHY USE PLANE WAVES

- SOLUTIONS TO WAVE EQUATION
- NATURAL BASIS SET TO USE FOR FOURIER ANALYSIS/SYNTHESIS

NOTE: PLANE WAVES NOT ALWAYS A GOOD APPROXIMANT (E.G. C NOT CONSTANT, NEARFIELD)

\[
\text{EX: } \left( \begin{array}{c}
(0) \\
(0) \\
(0) \\
\end{array} \right) \rightarrow \left( \begin{array}{c}
\text{Sensors} \\
\text{small aperture} \\
\text{compared to range} \\
\end{array} \right)
\]

\[
\text{NEARFIELD: PLANEWAVES NOT GREAT}
\]

\[
\text{FARFIELD: WAVEFRONTS ALMOST PLANAR ACROSS APERTURE}
\]
Think about spatial filtering: 2 element array

\[ \phi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} \]

Suppose we have a planewave incident on this array \( \theta = \theta_0, \phi = \pi/2 \)

Find difference in time of arrival between \( \phi_0 \) and \( \phi_1 \)

Remember \( c = \text{prop speed} \)
What is time difference?

\[ t_1 = \frac{d \cos \phi}{c} \]

Which sensor receives signal first?

Sensor 1

Does time delay change if azimuthal angle \( \phi \) changes?

Delay doesn't depend on \( \phi \)
Suppose that each sensor is corrupted by independent noise, e.g., flow noise in a sonar system model as temporally white and spatially independent.

How would you process these sensor data?

We know $\theta_0$.

$\Rightarrow$ Time align and average 2 signals.

Proposed scheme:

Sensor 0: $f(t, \theta_0)$

Sensor 1: $f(t, \phi_1) = f(t + T_1, \phi_0)$

Diagram:

1. Delay by $T_0 = 0$
2. Delay by $+T_1$
3. Average of 2 signals
This is the delay and sum beamformer. (Easily generalizes to \( n \) sensors)

To generalize, we need time delay to arbitrary sensor:

nth sensor at \( \mathbf{p}_n \)

Planewave coming from direction

\[
\mathbf{a} = -\mathbf{u}
\]

Unit vectors

\[
\mathbf{r}_n = \frac{\mathbf{a}^T \mathbf{p}_n}{c} = -\frac{\mathbf{u}^T \mathbf{p}_n}{c}
\]

\( \mathbf{a}^T \mathbf{p}_n \) = length of projection of \( \mathbf{p}_n \) onto \( \mathbf{a} \)
CHECK 1-d (2 ELEMENT CASE)

\[ p_1 = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} \quad a = -u = \begin{bmatrix} -\sin \Theta \cos \Phi \\ -\sin \Theta \sin \Phi \\ -\cos \Theta \end{bmatrix} \]

\[ \frac{a^T p_1}{c} = \frac{1}{c} (-d \cos \Theta) \quad \text{NEG. DELAY} \]
General plane-wave model for received signal at \( n \) sensors

The signal we observe at origin (if sensor there)

\[
f(t) \quad \leftrightarrow \quad F(w) = \text{Fourier X-form of time signal}
\]

Time signal at \( n \)th sensor

\[
f_n(t) = f(t - \tau_n)
\]

\[
F(w) e^{-jwT \tau_n} = F_n(w)
\]

\[
= F(w) e^{-j \pi T \beta n} = F(w) e^{-j \omega \left( \frac{-\pi \beta n}{c} \right)}
\]
Write Fourier X-form of all received signals in vector.

\[ F(\omega) = \begin{bmatrix} F_0(\omega) \\ F_1(\omega) \\ \vdots \\ F_{N-1}(\omega) \end{bmatrix} = F(\omega) \mathbf{v}_k(k) = F(\omega) \mathbf{v}_k(k) \]

\[ e^{-j k^T \mathbf{p}_0} \\ e^{-j k^T \mathbf{p}_1} \\ \vdots \\ e^{-j k^T \mathbf{p}_{N-1}} \]

\[ \mathbf{v}_k(k) = \text{array manifold vector} \]

\[ = \text{replica. vector} \]

Contains all spatial info.
NARROWBAND SIGNAL MODEL.

In real life we get this as follows:

\[ f_0(t) \xrightarrow{\text{BP filter at } w} F_0(w) \]

\[ f_1(t) \xrightarrow{\text{BP filter at } w} F_1(w) \]

\[ \vdots \]

\[ f_{N-1}(t) \xrightarrow{\text{BP filter at } w} F_{N-1}(w) \]

Ideally: Bandpass filter is infinitely narrow (passes 1 freq.)

In practice: Look at 1 bin of FFT
System implements NB spatial filtering.

When we do time-domain filtering, how we predict performance?

\[ \text{Freq. response} = H(w) \]

\[ e^{j\omega t} \xrightarrow{\text{LTI}} H(w) e^{j\omega t} \]
Spatial Filtering:

\[ v_k(k) \rightarrow \text{NB Spatial Filter (Linear)} \rightarrow I(\omega, k) \]

Arbitrary Planewave Vector

Frequency-Wavenumber Response

Complex Scalar

For our spatial filter:

\[ I(\omega, k) = w^H v_k(k) = \text{Complex Gain} \]

Response to arbitrary planewave

Complex gain given to a planewave of direction \( k \) and NB freq. \( \omega \)
If wave eqn is satisfied, $k$ can't be arbitrary. It is constrained $|k| = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

Possible $k$'s: $k = \frac{-2\pi}{\lambda} \ u(\theta, \phi)$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

Beam pattern is the freq-wave number response evaluated only at these possible angles:

$B(\omega; \theta, \phi) = |Y(\omega, k)|$

Spatial freq. resp. as fn of angle $k = \frac{-2\pi}{\lambda} u(\theta, \phi)$
Example: Linear array w/ equal spacing on z-axis

\[ p_{zn} = (n - (N - 1)d) \quad n = 0, \ldots, N-1 \]

\[ p_{xn} = p_{yn} = 0 \]

Array centered on origin only depends on \( k_z \)

Want: \( \mathbf{v}_k(k) = \mathbf{v}_k(k_z) = \begin{bmatrix} e^{j\frac{(N-1)}{2}k_zd} \\ \vdots \\ e^{-j\frac{(N-1)}{2}k_zd} \end{bmatrix} \]

\[ k_z = -\frac{2\pi}{\lambda} \cos \theta \]

\( u_z = \text{Directional Cosines} \)
\[ I(w, k) = w^H v_k(k) = w^H v_k(k_2) \]
\[ = \sum_{n=0}^{N-1} w_n e^{-j(n-(\frac{N-1}{2})) k_2 d} \]
\[ = e^{j(\frac{N-1}{2}) k_2 d} \sum_{n=0}^{N-1} w_n e^{-j n k_2 d} \]
\[ \text{Phase term} \]

**Define:** \( \Psi = -k_2 d \)

\[ I(w, \Psi) = e^{-j(\frac{N-1}{2}) \Psi} \sum_{n=0}^{N-1} w_n e^{-j n \Psi} \]

DTFT of \( w_n \)
Assume $w_n = \frac{1}{N}$  \hspace{1cm} \text{uniform weighting}

$$I(\omega, \Psi) = e^{-j \left( \frac{N-1}{2} \right) \Psi} \left( \sum_{n=0}^{N-1} \frac{1}{N} e^{-jn\Psi} \right)^N$$

\[\overset{\text{geometric series}}{=} \frac{1}{N} \sin \left( \frac{\Psi}{2} \right) \frac{1}{\sin \left( \frac{\Psi}{2} \right)}
\]

\[\overset{\text{DTFT}}{\frac{1}{N}} \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}
\]

$$I(\omega, k_2) = \frac{1}{N} \frac{\sin \left( \frac{N k_2 d}{2} \right)}{\sin \left( \frac{k_2 d}{2} \right)}$$
Sketch:

\[ |N \psi(\psi)| \]

\[ N = 10 \]

Zero crossings:

\[ \sin \left( \frac{N}{2} \psi \right) = 0 \]

\[ \frac{N}{2} \psi = \ell \pi \quad \text{multiple of } \pi \]

\[ \psi = \frac{2\pi \ell}{N} \]

Use l'Hôpital's to find HT at \( \psi = 0 \)

Periodic: repeats every

\[ \frac{\psi}{2} = \pi \ell \]

\[ \psi = 2\pi \ell \]
Sample in Time: Copies of Spectrum

Every $\Delta = \frac{2\pi}{T}$  \hspace{1cm} $T =$ Sample Period

Time Domain Problem: \hspace{1cm} $\omega = \Delta T$  \hspace{1cm} Normalized

$\uparrow$ \hspace{1cm} $\uparrow$

$\Delta T$ \hspace{1cm} $C T$

Freq. \hspace{1cm} Freq.

So then $\Delta T$ Spectrum Repeats Every $2\pi$

Sample in Space:

$k_z$-Space: Spectrum Repeats Every $\frac{2\pi}{d}$

$\Psi = -k_z d$  \hspace{1cm} Normalized Freq.

So $\Psi(\Psi)$ Repeats Every $2\pi$
\[ k_2 = \pm \frac{2\pi}{\lambda} \cos \theta \]

**MIN/MAX VALUES OF** \( k_2 \)

\[-\frac{2\pi}{\lambda} \leq k_2 \leq \frac{2\pi}{\lambda} \]

**VISIBLE REGION**

OF \( I(\omega, k_2) \)

**MIN/MAX VALUES OF** \( \psi \)

\[-\frac{2\pi d}{\lambda} \leq \psi \leq \frac{2\pi d}{\lambda} \]