**PLEASE TAKE 3 HANDOUTS**

**FILL OUT QUESTIONNAIRE**

**TOPICAL COVERAGE**

- Basics, intro to spatial filtering
- Deterministic spatial filtering
- Space-time random processes
- Statistical model
- Optimum beamforming
- Known statistics → performance limits
- Adaptive beamforming
- Estimate statistics from data

**COORDINATE SYSTEM**

\[ r = \sqrt{x^2 + y^2} \]

\[ \theta = \arctan \left( \frac{y}{x} \right) \]

\[ \phi = \text{azimuthal angle} \]

**TWO PROBLEMS OF INTEREST**

**SPATIAL FILTERING/BEAMFORMING**

**APPLICATION AREAS**

- Radar
- Sonar
- Ultrasound
- Seismology
- Hearing aids
- Communications

**IF WE WERE DOING TIME-DOMAIN PROCESSING**

**WHAT SET OF BASIS FUNCTIONS WOULD YOU USE?**

- Complex exponentials: \( e^{jut} \)
  - \( e^{jut} \) is eigenfunction of LTI system
  - \( H(w_0) \) is freq resp
  - Solutions to linear constant coefficient complex Diff Eqn.
  - \( c \) = foundation of Fourier analysis

**ASSUME A SEPARABLE SOLUTION:**

\[ f(x, y, z, t) = F(x) \cdot F(y) \cdot F(z) \cdot F(t) \]

For simplicity, assume

\[ f(x, y, z, t) = A e^{jut - jk_x x - jk_y y - jk_z z} \]

\( k_x, k_y, k_z, w \) are constants, \( w \geq 0 \)

**FOR SPATIAL FREQUENCIES ONLY**

\[ \frac{\partial^2}{\partial x^2} \left( r \right) = -k^2 A e^{jut - jk_x x - jk_y y - jk_z z} \]

**SPACE-TIME RANDOM PROCESSES**

**STATISTICAL MODEL**

**OPTIMUM BEAMFORMING**

**KNOWN STATISTICS → PERFORMANCE LIMITS**

**ADAPTIVE BEAMFORMING**

**ESTIMATE STATISTICS FROM DATA**

**BASIS FUNCTIONS → PLANEWAVES**

**MOTIVATION:** Planewaves as solutions to wave Eqn.

In free space, the scalar wave Eqn is:

\[ \nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \]

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \]

\( c \) = Acoustic or Electromagnetic

\( c \) = Propagation speed in medium assumed constant

**PARTIAL DIFFERENTIAL EQUATION**

**MONOCHROMATIC PLANEWAVE**

\[ j (ut - \mathbf{k} \cdot \mathbf{r}) \]

Monochromatic refers to time behavior of the signal, it's time signal with only 1 frequency.

\[ \mathbf{p} \]

\( \mathbf{k} \) = direction of propagation

Wavefronts, lines of constant phase, perpendicular to \( \mathbf{k} \)
**Constraint:** \[|k| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\omega}{c}\] (satisfies wave eqn.)

\[|k| = \frac{2\pi}{\lambda} = \frac{\omega}{c}\]

\(\lambda = \text{wave length} \quad \text{distance travelled in 1 temporal period}
\]

**Analog:** time domain

\[\Delta = \frac{2\pi}{T}\]

**Summary:** why use plane waves

- solve to wave eqn.
- natural basis set to use for Fourier analysis/synthesis

**Note:** plane waves not always a good approx (e.g. \(c\) not constant, nearfield)

**EX:** \(\frac{1}{\pi r^2}\) small aperture compared to range

**NAR: field:** waves fall off in near field: plane waves not great across aperture

**Can write** \(k\) as follows

\[k = \frac{2\pi}{\lambda} (-1)^m = -\frac{\omega}{c} \mathbf{a}\]

\(\mathbf{u} = \text{unit length} = \begin{bmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}\)

\(\mathbf{u} = \text{unit vector in direction specified by } \phi, \theta\)

**K = spatial freq; \(3-D\) space requires \(2\)-element vector**

\(\Rightarrow\) we've shown that monochromatic plane waves satisfy wave eqn.

**Think about spatial filtering:** 2 element array

**Suppose we have a plane wave incident on this array**

**Remember** \(c = \text{prop. speed}\)

**What is time difference?**

\[t_1 = \frac{d \cos \theta_0}{c}\]

**Which sensor receives signal first?**

Sensor 1

**Does time delay change if azimuthal angle \(\phi\) changes?**

Delay doesn't depend on \(\phi\)

**This is the delay and can be beamformed.** (easily generalizes to \(N\) sensors)

To generalize, we need time delay to arbitrary sensor:

Nth sensor at \(\phi_n\)

Plane wave coming from direction

\[\mathbf{u} = \begin{bmatrix} \sin\theta \cos\phi_n \\ \sin\theta \sin\phi_n \\ \cos\theta_n \end{bmatrix}\]

\[\mathbf{u} = \text{unit vectors}\]

**\(T_n = \frac{\mathbf{u}^T \mathbf{p}_1}{c} = - \frac{\mathbf{u}_0^T \mathbf{p}_1}{c}\)**

\(\Rightarrow\) length of protection of \(\phi_n\) onto \(\phi_0\)

If we wanted to represent signal \(u\): more complicated temporal structure

\(\Rightarrow\) use Fourier

**Want pulse:**

\[f(t) = \int f(u) e^{-jut} du\]

**Fourier synthesis**

\[f(u) = \text{periodic}, \text{ use Fourier series...}\]

Do same thing in spatial domain:

\[f(t, \phi) = \int \int f(u, \mathbf{p}) e^{-j(ut - \mathbf{u}^T \mathbf{p})} du d\mathbf{p}\]

\[f(u, \mathbf{p}) = \int f(t, \phi) e^{j(ut - \mathbf{u}^T \mathbf{p})} dt d\phi\]
General plane-wave model for received signal at $N$ sensors

The signal we observe at origin if sensor there:

$$ F(t) \leftrightarrow F(w) = \text{Fourier X-form of time signal} $$

Time signal at $n$th sensor:

$$ f_n(t) = f(t - \tau_n) = F(w) e^{-j\omega \tau_n} = F(w) e^{-j\omega \tau_n} \left( \frac{e^{-j\omega \tau_n}}{e^{-j\omega \tau_n}} \right) $$

Linear processor:

- Complex scalar weights $X = \text{Conjugate}$
- Hermitian transpose $H = \text{Hermitian transpose}$

System implements NB spatial filtering.

Spatial filtering:

- $X_K(\omega) = \text{NB spatial filter (linear) frequency-wavenumber response}$
- Complex scalar vector

For our spatial filter:

$$ X_K(\omega) = W_H Y_K(\omega) $$

Example: Linear array w/ equal spacing on $z$-axis

**Array centered on origin:**

$$ k_x = \frac{\omega}{c} \cos \theta, \quad k_y = 0, \quad k_z = \frac{\omega}{c} \sin \theta $$

$$ X_K(\omega) = \sum_{n=0}^{N-1} w_n e^{-j(\omega - \omega_n)k_0d} $$

**Define:** $\Psi = -k_0d$

$$ X(\omega, k) = e^{-j\Psi} \sum_{n=0}^{N-1} w_n e^{-j\omega_kd} $$

**Assume $w_n = \frac{1}{N}$ uniform weighting:**

$$ X(\omega, k) = \sum_{n=0}^{N-1} \frac{1}{N} e^{-j\omega_kd} $$

$$ X(\omega, k) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega_kd} $$

$$ X(\omega, k) = \frac{1}{N} \sin \left( \frac{\omega_kd}{2} \right) \sin \left( \frac{k\omega_d}{2} \right) $$
Sample in time: copies of spectrum every \( \Delta \tau = \frac{2\pi}{T} \), \( T = \text{sampling period} \)

Time domain problem: \( \omega = N T \), normalized freq.

So the DT spectrum repeats every \( 2\pi \)

Sample in space:

- \( k_x \)-space: spectrum repeats every \( \frac{2\pi}{d} \)
- \( \Psi = -k_x d \) \( \Rightarrow \) normalized freq.

So \( \mathcal{I}(\Psi) \) repeats every \( 2\pi \)

\[ k_x = -\frac{2\pi}{\lambda} \cos \Theta \]

Min/max values of \( k_x \):

\( -\frac{2\pi}{\lambda} \leq k_x \leq \frac{2\pi}{\lambda} \) (visible region of \( I(x,k_x) \))

Min/max values of \( \Psi \):

\( -\frac{2\pi d}{\lambda} \leq \Psi \leq \frac{2\pi d}{\lambda} \)