WARM-UP PROBLEM

GIVEN: \( w[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{other} \end{cases} \)

FIND DTFT OF \( w[n] \) + SKETCH \( |W(c^jw)| \)

FIND z-TRANS. OF \( w[n] \)

SKETCH POLES + ZEROS OF \( W(z) \)
WARM-UP

\[ W(z) = \sum_{n} w[n] z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}} \]

\[ W(e^{j\omega}) = \sum_{n} w[n] e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \]

**? How do \( z \) and \( \text{FOURIER} \) relate?**

\[ W(e^{j\omega}) = W(z) \bigg|_{z = e^{j\omega}} \quad \text{\( z \) on unit circle} \]

Diagram showing the magnitude of \( W(e^{j\omega}) \) for different \( \omega \) values.
W(z) = \frac{1 - z^{-N}}{1 - z^{-1}} \cdot \frac{z^N}{z^N}

= \frac{z^{N-1}}{z^{N-1}(z-1)}

Zeroes: 
\[ z^N = 1 = e^{j2\pi n} \]
\[ z = e^{j\frac{2\pi n}{N}} \]

Poles: 
\[ z^{N-1} = 0 \]
\[ z = 0 \quad N-1 \text{ poles at } z = 0 \]
1 pole at \( z = 1 \)
CH. 3: DESIGN PROBLEM

choose \( w \) to meet certain criteria

\[ \text{ex. minimize beamwidth} \]

\[ \text{(filter passband width)} \]

control sidelobe levels

uncertainty principle:

\[ \sqrt{\Delta \theta^2} \sqrt{\Delta k^2} \geq \frac{1}{2} \]

normalized mean sq. width of aperture

normalized mean sq. width of spatial response
Focus on linear, equally spaced arrays

Design methods:

- $w = \text{standard window as used in spectral estimation}$
- Frequency sampling
- Minimize BW for fixed sidelobe level $\Rightarrow$ Dolph-Chebychev
- Least squares
  - Minimize MSQ error between desired + actual beam pattern
- Min-Max (Parks-McClellan)
On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform

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Abstract—This paper makes available a concise review of data windows and their affect on the detection of harmonic signals in the presence of broad-band noise, and in the presence of nearby strong harmonic interference. We also call attention to a number of common errors in the application of windows when used with the fast Fourier transform. This paper includes a comprehensive catalog of data windows along with significant performance parameters from which the different windows can be compared. Finally, an example demonstrates the use and value of windows to resolve closely spaced harmonic signals characterized by large differences in amplitude.

I. INTRODUCTION

There is much signal processing devoted to detection and estimation. Detection is the task of determining if a specific signal is present in an observation, while estimation is the task of obtaining the values of the parameters describing the signal. Often the signal is complicated or is corrupted by interfering signals or noise. To facilitate the detection and estimation of signal sets, the observation is decomposed by a basis set which spans the signal space [1]. For many problems of engineering interest, the class of signals being sought are periodic which leads quite naturally to a decomposition by a basis consisting of simple periodic functions, the sines and cosines. The classic Fourier transform is the mechanism by which we are able to perform this decomposition.

By necessity, the observed signal we process must be of finite extent. The extent may be adjustable and selectable, but it must be finite. Processing a finite-duration observation imposes interesting and interacting considerations on the harmonic analysis. These considerations include detectability of tones in the presence of nearby strong tones, resolvability of similar-strength nearby tones, resolvability of shifting tones, and biases in estimating the parameters of any of the aforementioned signals.

For practicality, the data we process are $N$ uniformly spaced samples of the observed signal. For convenience, $N$ is highly composite, and we will assume $N$ is even. The harmonic estimates we obtain through the discrete Fourier transform (DFT) are $N$ uniformly spaced samples of the associated periodic spectra. This approach is elegant and attractive when the processing scheme is cast as a spectral decomposition in an $N$-dimensional orthogonal vector space [2]. Unfortunately, in many practical situations, to obtain meaningful results this elegance must be compromised. One such compromise consists of applying windows to the sampled data set, or equivalently, smoothing the spectral samples. The two operations to which we subject the data are sampling and windowing. These operations can be performed in either order. Sampling is well understood, windowing is less so, and sampled windows for DFT's significantly less so! We will address the interacting considerations of window selection in harmonic analysis and examine the special considerations related to sampled windows for DFT's.

II. HARMONIC ANALYSIS OF FINITE-EXTENT DATA AND THE DFT

Harmonic analysis of finite-extent data entails the projection of the observed signal on a basis set spanning the observation interval [1], [3]. Anticipating the next paragraph, we define $T$ seconds as a convenient time interval and $NT$ seconds as the observation interval. The sines and cosines with periods equal to an integer submultiple of $NT$ seconds form an orthogonal basis set for continuous signals extending over $NT$ seconds. These are defined as

$$
\begin{align*}
\cos \left[ \frac{2\pi}{NT} kt \right] & \quad k = 0, 1, \cdots, N-1, N, N+1, \cdots \\
\sin \left[ \frac{2\pi}{NT} kt \right] & \quad 0 \leq t < NT.
\end{align*}
$$

We observe that by defining a basis set over an ordered index $k$, we are defining the spectrum over a line (called the frequency axis) from which we draw the concepts of bandwidth and of frequencies close to and far from a given frequency (which is related to resolution).

For sampled signals, the basis set spanning the interval of $NT$ seconds is identical with the sequences obtained by uniform samples of the corresponding continuous spanning set up to the index $N/2$,

$$
\begin{align*}
\cos \left[ \frac{2\pi}{NT} kNt \right] &= \cos \left[ \frac{2\pi}{N} kn \right] \quad k = 0, 1, \cdots, N/2 \\
\sin \left[ \frac{2\pi}{NT} kNt \right] &= \sin \left[ \frac{2\pi}{N} kn \right] \quad n = 0, 1, \cdots, N-1.
\end{align*}
$$

We note here that the trigonometric functions are unique in that uniformly spaced samples (over an integer number of periods) form orthogonal sequences. Arbitrary orthogonal functions, similarly sampled, do not form orthogonal sequences. We also note that an interval of length $NT$ seconds is not the same as the interval covered by $N$ samples separated by intervals of $T$ seconds. This is easily understood when we
Freq.-Sampling for uniformly-spaced arrays

Recall: $B(\psi) = I(\psi) = e^{-j\left(\frac{N-1}{2}\right)\psi} \left( \sum_{n=0}^{N-1} w_n e^{-jn\psi} \right)$

DTFT of $w_n$ (.weight sequence)

Array centered at $z = 0$

Rearrange:

$B^*(\psi) e^{-j\left(\frac{N-1}{2}\right)\psi} = \sum_{n=0}^{N-1} w_n e^{-jn\psi}$
DTFT's are periodic with period $2\pi$.

Can we get $w_n$ by sampling this desired DTFT? Yes $\rightarrow$ DFT.

DFT: Samples equally spaced 0 to $2\pi$.

Van Trees: $N$ equally spaced samples from $-\pi$ to $+\pi$.

$$\psi_k = (k - (\frac{N-1}{2})) \frac{2\pi}{N} \quad k = 0, \ldots, N-1$$
\[ B(k) = B^*(\psi_k)e^{-j\left(\frac{N-1}{2}\right)\psi_k} = \sum_{n=0}^{N-1} w_n e^{-jn(\psi_k - (\frac{N-1}{2})\pi)} \]

\[ B(k) = \sum_{n=0}^{N-1} w_n e^{-jn(\frac{N-1}{2})\pi - jnk\frac{2\pi}{N}} \]

\[ B(k) = \sum_{n=0}^{N-1} w_n e^{-j\frac{2\pi}{N}nk} \]

\[ \text{Looks like DFT!} \]
GIVEN: \( B(k) \) \hspace{1cm} \text{SAMPLES OF DESIRABLE PATTERN} \hspace{1cm} -j n \left( \frac{n-1}{N} \right) \pi \\

\text{WANT:} \hspace{0.5cm} b_n \rightarrow w_n = b_n e^{-j n \left( \frac{n-1}{N} \right) \pi} \\

\text{IDFT:} \hspace{0.5cm} b_n = \frac{1}{N} \sum_{k=0}^{N-1} B(k) e^{j \frac{2\pi n k}{N}} \\

\text{WILL GET} \hspace{0.5cm} B(y) = B_{\text{desired}}(y) \bigg|_{y = y_k} \\

\text{COMPUTE} \hspace{0.5cm} B(y) = w^H y(y) \hspace{1cm} \text{SAMPLE POINT} \hspace{1cm} \text{FOR ALL } y \hspace{0.5cm} \text{TO GET PERFORMANCE}
Minimum Beamwidth for Specified Sidelobe Level

Dolph-Chebyshev Approach

Want:

\[ B(\psi) \]

\[ B(\psi) \text{ is real and even} \]

\[ B(\psi) = w^H v(\psi) = \sum_{n=0}^{N-1} w_n^* e^{-j\psi(n-(\frac{N-1}{2}))} \]
If \( N = 0, 0, 0, 0 \) \( M = n \frac{N-1}{2} \) 
\[ B(\psi) = \sum_{m=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} a_m e^{j\psi m} \]
\[ a_m = w_n \quad n = m + \frac{N-1}{2} \]

Assuming \( a_m \) Real and Even

\[ B(\psi) = a_0 + 2 \sum_{m=1}^{\frac{N-1}{2}} a_m \cos \left( m \psi \right) \]

If \( N = \text{Even} \)

\[ B(\psi) = 2 \sum_{m=1}^{\frac{N}{2}} a_m \cos \left( \left( m - \frac{1}{2} \right) \psi \right) \]
REWRITING:

**ODD**  
$$B(y) = \sum_{k=0}^{\frac{n-1}{2}} \alpha_k \cos \left(2k \frac{y}{2} \right)$$

**EVEN**  
$$B(y) = \sum_{k=1}^{\frac{n}{2}} \alpha_k \cos \left((2k-1) \frac{y}{2} \right)$$

$$\alpha_k = \begin{cases} 
2\alpha_k & k \neq 0 \\
\alpha_k & k = 0
\end{cases}$$

DOLPH-CHEBYCHEV:

$B(y)$ AS POLYNOMIAL IN $\cos(\frac{y}{2})$ OF ORDER $N-1$
Look at term like \( \cos \left( M \frac{\psi}{2} \right) \)

\[
\cos \left( M \frac{\psi}{2} \right) = \text{Re} \left( \left( e^{j \frac{\psi}{2}} \right)^M \right)
\]

\[
= \text{Re} \left\{ \left[ \cos \left( \frac{\psi}{2} \right) + j \sin \left( \frac{\psi}{2} \right) \right]^M \right\}
\]

\[\text{Binomial Thm!}\]

\[
(a+b)^M = \sum_{\ell=0}^{M} \binom{M}{\ell} a^{M-\ell} b^\ell
\]

\[
\binom{M}{\ell} = \frac{M!}{\ell! (M-\ell)!}
\]
\[ m = 2 \]

\[
\cos(2(\frac{\psi}{2})) = \Re \left\{ \left[ \cos(\frac{\psi}{2}) + j \sin(\frac{\psi}{2}) \right]^2 \right\}
\]

\[
= \Re \left\{ \cos^2(\frac{\psi}{2}) + 2j \cos(\frac{\psi}{2})\sin(\frac{\psi}{2}) - \sin^2(\frac{\psi}{2}) \right\}
\]

\[
= \cos^2(\frac{\psi}{2}) - \sin^2(\frac{\psi}{2})
\]

\[
= \cos^2(\frac{\psi}{2}) - (1 - \cos^2(\frac{\psi}{2}))
\]

\[
= 2\cos^2(\frac{\psi}{2}) - 1 = \cos(2(\frac{\psi}{2}))
\]

\[ B(\psi) = \text{N-1 ORDER POLYNOMIAL IN POWERS OF } \cos\left(\frac{\psi}{2}\right) \]
\[
\cos(n \frac{\pi}{2}) = T_n(x) \bigg|_{x = \cos\left(\frac{\pi}{2}\right)}
\]

\[T_n \triangleq \text{Mth CHEBYCHEV POLYNOMIAL}\]

\[T_0(x) = 1\]

\[T_1(x) = x\]

\[T_2(x) = 2x^2 - 1\]

\[T_3(x) = 4x^3 - 3x\]

\[\vdots\]

\[\vdots\]

\[\text{RECURSION:} \quad n \geq 2\]

\[T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)\]
\[ T_M(x) = \begin{cases} \cos(M \cos^{-1}(x)) & |x| \leq 1 \\ \cosh(M \cosh^{-1}(x)) & x > 1 \\ (-1)^m \cosh(M \cosh^{-1}(x)) & x < -1 \end{cases} \]

**Properties:**

\* \( T_M(x) \) has \( M \) real roots in the interval \( |x| < 1 \)

\[ T_M(x) = \cos(M \frac{\psi}{2}) \bigg|_{\cos\left(\frac{\psi}{2}\right) = x} \]

**Roots:** \( \cos(M \frac{\psi}{2}) = 0 \)

\[ M \frac{\psi}{2} = \frac{\pi}{2} (2p - 1) \quad p = 1, \ldots, M \]
ROUTES IN Y SPACE:

\[ \psi = \pi \left( \frac{2p-1}{M} \right) \]

IN X -SPACE ROOTS ARE:

\[ x_p = \cos \left( (2p-1) \frac{\pi}{2M} \right) \]

PROPERTIES:

* \( T_m(x) \) HAS ALTERNATING MINIMA AND MAXIMA

\[ |T_{\text{MIN}}| = |T_{\text{MAX}}| = 1 \]

EQUI RIPPLE IN X VARIABLE

* AT \( x = \pm 1 \), \( |T_m(x)|_{x=\pm 1} = 1 \)

\( x > 1 \) \( |T_m(x)| > 1 \)
**Dolph-Cheby Idea:**

- \( B(y) \) is an \( N-1 \) order polynomial in \( \cos \left( \frac{y}{2} \right) \)

- Set \( B(y) = \text{Chebychev polynomial} \) so we get the equiripple property

Define a mapping so that \( B(y) \) maximum (M.L. Max) corresponds to a pt \( x_0 \geq 1 \)

\[
R = \frac{\text{M.L. Max}}{\text{sidelobe level}}
\]

Solve for \( x_0 \):

\[
T_{N-1}(x_0) = R = \cosh \left( (N-1) \cosh^{-1}(x_0) \right)
\]
Figure 3.2.4
\[ x_0 = \cosh \left( \frac{1}{N-1} \cosh^{-1}(R) \right) \]

**RECALL MAPPING:**
\[ x = \cos \left( \frac{\psi}{2} \right) \]

**RESTRICTS** \(-1 \leq x \leq +1\)

**ADJUST MAPPING:**
\[ q = \frac{x}{x_0} = \cos \left( \frac{\psi}{2} \right) \quad \left( q = \frac{w}{\text{in TEXTBOOK}} \right) \]
\[ x = x_0 \cos \left( \frac{\psi}{2} \right) \]

**BEAM PATTERN:**
\[ B(\psi) = \frac{1}{R} T_{N-1}(x_0 \cos(\frac{\psi}{2})) \]
How to find weights given $b(y) = \text{Cheby poly.}$

\[ w^H v(\psi) \bigg|_{\psi=0} = 1 \]

Mainlobe constraint

\[ w^H v(\psi_p) = 0 \]

$N-1$ zeros

(Nulls in beam pattern)

Matrix form:

\[ w^H V = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \]

Or

\[ V^H w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

\[ V = \begin{bmatrix} v(0) & v(\psi_1) & \cdots & v(\psi_p) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \]
SOLVING:

\[ w = (V^H)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

_TEXT CALLS THIS \( \varphi_1 \)

KNOW: WHERE ARE \( \psi_p \)'s? NULLS

ORIGINAL SPACE: \( \psi = \frac{\pi}{N-1} (2p-1) \) \( p=1, \ldots, M \)

\( x_p = \cos \left( \frac{\pi}{2} \left( \frac{2p-1}{N-1} \right) \right) \)

NEW SPACE: \( q_p = \frac{x_p}{x_0} = \frac{1}{x_0} \cos \left( \frac{\pi}{2} \left( \frac{2p-1}{N-1} \right) \right) \)
New zeros:  \( \psi_p = 2\cos^{-1}(q_p) \)

\[ \psi_p = 2\cos^{-1}\left( \frac{1}{x_0} \cos\left( \frac{\pi}{2} \left( \frac{2p-1}{N-1} \right) \right) \right) \quad p = 1, \ldots, N-1 \]

**Mapping:**

\[ B(\psi) = \frac{1}{R} T_{N-1}\left( x_0 \cos\left( \frac{\psi}{2} \right) \right) = T_{N-1}(x) \]

\[ x = x_0 \cos\left( \frac{\psi}{2} \right) \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{2\pi d}{x} )</th>
<th>( \frac{-2\pi d}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi = \frac{2\pi}{\lambda} d \cos \theta )</td>
<td>( \frac{2\pi d}{\lambda} )</td>
<td>0</td>
<td>( \frac{-2\pi d}{\lambda} )</td>
<td>( x_0 \cos\left( \frac{\pi d}{\lambda} \right) )</td>
<td>( x_0 \cos\left( \frac{\pi d}{\lambda} \right) )</td>
</tr>
<tr>
<td>( x = x_0 \cos\left( \frac{\psi}{2} \right) )</td>
<td>( x_0 \cos\left( \frac{\pi d}{\lambda} \right) )</td>
<td>( x_0 )</td>
<td>( x_0 \cos\left( \frac{\pi d}{\lambda} \right) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.26: Dolph–Chebychev results for N=8
\[ d = \frac{A}{4} : \quad x_0 \cos \left( \frac{\pi \sqrt{2}}{x} \right) \leq x \leq x_0 \]

\[ x_0 \cos \left( \frac{\pi}{4} \right) \leq x \leq x_0 \]

\[ 0.707 x_0 \leq x \leq x_0 \]

\[ d = \frac{\lambda}{2} \quad x_0 \cos \left( \frac{\pi \sqrt{2}}{\lambda} \right) \leq x \leq x_0 \]

\[ 0 \leq x \leq x_0 \]

\[ d = \frac{3\lambda}{4} \quad -0.707 x_0 \leq x \leq x_0 \]

\[ d \leq \frac{\lambda}{2} \cos^{-1} \left( \frac{1}{x_0} \right) \quad \text{FOR MAPPING TO BE VALID} \]

\[ \text{LEFT EDGE} = -1 \]
Riblet - Chebychev

- Uses expansion in terms of $\cos(y)$ instead $\cos\left(\frac{y}{2}\right)$
- Polynomial of order $\frac{n-1}{2}$ (n odd)
- Mapping: $x = c_1 \cos y + c_2$
- Solve for $c_1 + c_2$ so that full range of Chebychev polynomial is used

$-1 \leq x \leq x_0$

$d = \frac{\lambda}{2} \text{ No difference} \quad | \quad d < \frac{\lambda}{2} \text{ Difference}$
VILLENEUVE

• START W/ ZEROS ASSOCIATED W/ UNIFORM WEIGHTING (WARM-UP)

• EXCHANGE 2^n OF THOSE NEAR BROADSIDE FOR ~CHEBYSHEV ROOTS

=> RESULT: BETTER S.L. NEAR BROADSIDE DECRAISING S.L. FARTHER OUT
Figure 3.30
Roots in z-plane, N=21, n=6

Real

Imaginary

Uniform roots
Villeneuve roots
Chebychev roots