Approximate mode filtering

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November 2004

Proceedings of the 38th Asilomar Conference on Signals, Systems, and Computers, pp. 1436-1440
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Abstract—Normal modes, the eigenfunctions of the ocean waveguide, are useful in matched field processing and acoustic tomography applications. Mode filter design typically requires exact knowledge of the ocean environment, i.e., the sound speed profile. This paper describes a technique for designing mode filters using modeshapes derived from a uniform WKB-like approximation. The errors introduced by using the approximate modeshapes are analyzed.

I. INTRODUCTION

Normal modes are the eigenfunctions of the ocean waveguide. Modal decompositions of signals are useful in acoustic tomography and source localization applications, e.g., see [1] and [2]. Since the signals propagating in different modes often overlap in time, they must be resolved using spatial processing. Standard techniques exist for estimating the mode shapes from vertical line array (VLA) measurements. These techniques assume that the mode shapes and wavenumbers are known. The modes depend on the ocean environment, particularly on the sound speed profile (SSP). In realistic experimental scenarios, the SSP is often not known exactly. For example in both the 1998 North Pacific Acoustic Laboratory (NPAL) [3] experiment and the ongoing SPICE04 experiment, the environment was sampled along the VLA span, but not above and below the array.1 To facilitate mode processing, an approximation to the modeshapes is needed that only requires the environmental information along the array. Ideally, the approximation should provide an analytical form for the modeshapes, rather than just a numerical solution. An analytical form is desirable because it provides more intuition about the filtering problem and may permit simpler design of robust mode filters.

WKB theory provides the most common approximate solution to the mode problem. The main problem with using the standard WKB solution to design mode filters is that it is singular at the turning points. Miller and Good developed a uniform WKB-like solution to the Schrödinger equation that is continuous at the turning points [4]. The remainder of this paper applies Miller and Good’s approach to generate modes for the ocean waveguide and examines the characteristics of the resulting approximate mode filters. Specifically, Section II defines the mode filtering problem and several useful performance metrics. Section III describes Miller and Good’s approximate method. To evaluate the usefulness of the approximate solution, Section IV presents a series of examples for three deep water waveguides. Section V concludes the paper.

II. MODE FILTERING

The normal modes are derived from the frequency domain wave equation (Helmholtz equation) [5]. At each frequency, a mode is characterized by its wavenumber \( k_m \) and its modeshape \( \phi_m(z) \). For a given environment, defined by the sound speed profile and boundary conditions, the modes satisfy a second-order eigenvalue equation, e.g., in cylindrical coordinates (assuming constant density):

\[
\frac{d^2 \phi_m(z)}{dz^2} + k^2(z)\phi_m(z) = 0,
\]

where

\[
k(z) = \frac{\omega}{c(z)}, \quad k\omega = \sqrt{k^2(z) - k_m^2}.
\]

\( \omega \) is the temporal frequency, \( c(z) \) is the sound speed as a function of depth \( z \), and \( k(\omega, z) \) is the medium wavenumber. The modal wavenumber determines propagation characteristics, such as phase and group speeds, and the modeshape determines the spatial distribution of pressure due to each mode. The modeshapes form an orthogonal basis set. Fig. 1 shows the shapes of the first 10 modes for a measured deep water environment near Hawaii.

Since the modes are an orthonormal set, the narrowband pressure field \( p(z) \) at frequency \( \omega \) and depth \( z \) can be written as a weighted sum of modes,

\[
p(\omega, z) = \sum_{m=1}^{M} a_m \phi_m(z) + n(z),
\]

where \( a_m \) is the coefficient for the \( m \)th mode and \( n(z) \) is additive noise. The goal of mode processing is to estimate the \( a_m \) coefficients from measurements of the pressure field along a VLA. For example, Fig. 1 shows the hydrophone locations for a 40-element VLA designed to resolve the first 10 modes of that environment. In vector notation, the VLA measurements can be written

\[
p = Ea + n,
\]

where \( E \) is the matrix of sampled modeshapes, \( a \) is the vector of mode coefficients, and \( n \) is the vector of observation

This work was funded by an ONR Ocean Acoustics Entry-Level Faculty Award.

1In these experiments, data recorders mounted on the VLA provide a year-long record of temperature from which sound speed can be estimated.
III. APPROXIMATE MODE SOLUTIONS

With the exception of very simple environments, such as an isovelocity shallow water channel, obtaining the modes requires numerical solution of Eq. 1. Underwater acoustic mode codes such as Kraken [11] and Pruefer [6] can easily generate the modes given the complete SSP and the boundary conditions. When the environment is not known exactly, an approximate solution can be used. As noted in the Introduction, WKB theory provides the most commonly-used approximate solution to the mode problem. Many mathematics texts, such as Bender and Orszag [12], discuss WKB theory. For specific information on the application of WKB to the underwater acoustic mode problem, see Jensen et al. [13]. In the WKB approximation, the mode eigenvalues satisfy the following equation

\[ \int_{z_l}^{z_u} k_z(z) dz = \left( m - \frac{1}{2} \right) \pi, \]

where \( z_l \) and \( z_u \) are the lower and upper turning depths, respectively. The WKB solution to the modeshape is

\[ \phi(z) \approx A e^{i \int k_z(z) dz / \sqrt{k_z(z)}} + B e^{-i \int k_z(z) dz / \sqrt{k_z(z)}}, \]

where \( A \) and \( B \) are scaling constants. The advantage of the WKB solution is that it only requires knowledge of the wavenumber \( k_z(z) \) (hence knowledge of \( k(z) \) and \( c(z) \)), over the depths where the modeshapes are needed, i.e., over the span of the VLA. The main disadvantage of the WKB solution is that it is singular at the mode turning points, i.e., where the solution changes from an oscillatory one to a decaying exponential. There are methods for obtaining the modeshape near the singular point [12], but the solution must be patched together across different regions using connection formulas. The WKB solution does not provide a uniformly-valid solution that is a continuous function of depth.

In 1953 Miller and Good published a WKB-like solution to the one-dimensional Schrödinger equation that is continuous at the two turning points [4]. Their approach consists of transforming the problem in such a way that the mode \( \phi(z) \) can be written in terms of the corresponding mode \( \psi(x) \) of the harmonic oscillator problem, i.e.,

\[ \phi(z) = (x')^{-1/2} \psi(x(z)), \]

where

\[ \frac{d^2 \psi}{dx^2} + (E - x^2) \psi = 0. \]

Essentially the problem reduces to finding the modes of the harmonic oscillator (whose solutions are well-known) and determining an appropriate mapping between the depth co-

ordinates \( z \) and \( x \) for each problem. A proper selection of the \( z \leftrightarrow x \) mapping can guarantee the continuity of the modeshapes. Miller and Good choose \( x(z) \) such that a turning point in the original problem maps to a turning point in the harmonic oscillator problem. The following equation defines

\[ \text{Output Mode} \]

\[ \text{Input Mode} \]

\[ \text{dB} \]

\[ \text{Speed (m/s)} \]

\[ \text{Depth (m)} \]

\[ \text{SSP} \]

\[ \text{Mode number} \]

\[ \text{Modeshapes for measured SSP near Hawaii} \]

\[ \text{Fig. 1. Sound speed profile and first 10 modes for a measured environment near Hawaii. The waveguide is approximately 5400 m deep, but only the top 2500 m are shown. Modes were computed using Baggeroer's Pruefer normal mode program [6]. The crosses on the right indicate the hydrophone locations for a 40-element VLA designed to resolve the first 10 modes.} \]

\[ \text{Fig. 2. Beampattern for a 10-mode PI filter computed using the exact modeshapes shown in Fig. 1.} \]

noise. Standard methods exist for estimating the modal content of arriving signals. The two most common spatial filters for modes are the matched filter [7], [8] and the pseudo-

inverse (PI) filter [9]. This paper focuses on the PI filter, which computes the vector \( \hat{a} \) of estimated mode coefficients as follows:

\[ \hat{a} = (E^T E)^{-1} E^T p. \]
this mapping
\[
\int_{z_l}^{z} k_z(z) \, dz = \int_{\sqrt{E}}^{x} (E - \sigma^2) \, d\sigma,
\]
where \( z_l \) is the lower turning depth for the \( m \)th mode in the original problem. With this choice, the eigenvalue condition for Miller and Good’s approach is identical to the WKB eigenvalue condition given in Eq. 6. The resulting approximate modeshape for the \( m \)th mode can be written
\[
\phi_m(z) \approx A_m \frac{e^{-z^2(z)/2} H_{m-1}(x(z))}{\sqrt{x'(z)}},
\]
where \( A_m \) is a scale factor, and \( H_m \) is the \( m \)th Hermite polynomial.

Miller and Good’s approach is directly applicable to the ocean acoustic mode problem, as discussed at the Underwater Acoustic Signal Processing Workshop [14]. In practice, the integrals over \( k_z(z) \) in Equations 6 and 10 must be computed numerically. The integral on the right side of Eq. 10 can be computed analytically. To facilitate the derivative calculation required in Eq. 11, a polynomial fit to \( x(z) \) is first computed given values obtained from Eq. 10.

Recent work by Grigorieva and Fridman indicates that an approximate mode solution similar to Miller and Good’s was developed by Buldyrev and Slavjanaov [15]. Grigorieva and Fridman use the approximation to study the propagation of axial waves over long ranges [16], [17].

IV. EXAMPLES

This section illustrates the application of Miller and Good’s method through a series of examples for the three deep water environments defined in Sec. IV-A. Sec. IV-B compares the approximate modes with the exact modes for the three environments, and Sec. IV-C evaluates the accuracy of mode filters designed with the approximate modeshapes.

A. Deep water environments

Fig. 3 shows three representative SSP’s for a deep water channel. In deep water there is typically a minimum in sound speed between 700 m and 1000 m, and the speed increases towards the bottom and the surface. The green dash-dot curve shows the “deep six” analytical profile defined by Miller [18]:
\[
c(z) = \bar{c} \left( 1 + \frac{c_0}{2} \left[ \frac{\eta}{1 - \eta/6} \right] \right),
\]
where
\[
\eta = \frac{2}{B} (z - \bar{z}).
\]
\( \bar{c} \) is the axial (minimum) sound speed, \( \bar{z} \) is the depth of the sound channel axis, and \( c_0 \) and \( B \) are known constants. The red dashed curve in Fig. 3 shows the profile corresponding to a CTD measurement of an environment near Hawaii. The blue solid curve shows an archival profile for the same location generated using temperature and salinity profiles from the World Ocean Atlas (WOA) [19], [20].

B. Approximate wavenumbers and modeshapes

The accuracy of the Miller/Good solution can be measured by comparing the approximate modes to the exact solutions computed using a standard mode code such as Pruefer [6]. Fig. 4 shows the wavenumber errors for the first 20 modes of the three deep water environments. To put this result in perspective it is important to note that the exact wavenumbers lie between 0.315 and 0.318, thus the maximum error is on the order of 0.004 percent. The Deep-6 profile has the smallest error, which is not surprising given that it the type of profile that the Miller/Good method was designed for, i.e., it is a smooth profile with sound speed that is strictly increasing away from the channel axis.

Fig. 5 compares the approximate modeshapes for the three environments with the corresponding exact modeshapes. The agreement is quite good. Fig. 6 zooms in on the first five modeshapes to illustrate the small differences. The Deep-6
approximate modes appear to line up perfectly with the exact modes, whereas the modes for the two other profiles show small perturbations away from the exact shapes.

C. Approximate mode filter characteristics

Modeshape errors are a form of mismatch. To evaluate how much the errors in the approximate modeshapes affect the PI mode filter, consider the beampattern for the mismatched case, i.e.,

$$\text{Beampattern} = 20 \log_{10} (W_{\text{approx}}^T E_{\text{exact}})$$  \hspace{1cm} (14)

$W_{\text{approx}}$ is the mode filter designed with the approximate modeshapes, and $E_{\text{exact}}$ is the matrix of exact modeshapes. Fig. 7 shows the mismatched beampatterns for the three environments. Recall that if the exact modeshapes are used, there should be no crosstalk among the 10 modes included in these PI filters. Based on Fig. 7, the modeshape errors for the Deep-6 case are negligible because the beampattern for the approximate mode filter retains its diagonal structure. The off-diagonal terms are less than -48 dB. For the CTD and WOA profiles, the modeshape errors result in peak crosstalk on the order of -18 dB and -21 dB, respectively.

To put the approximate mode filter results in perspective, it is useful to consider the amount of potential mismatch due to uncertainty in the profile. The ocean is a dynamic medium. Internal waves with time scales on the order of minutes to hours cause fluctuations in the sound speed. Fig. 8 shows one realization of the WOA archival profile perturbed by internal waves. The internal waves were generated using the method of Colosi and Brown [21]. Note that the features in the internal wave profile are qualitatively similar to those observed in the measured (CTD) profile. Fig. 9 compares the beampattern for an exact PI filter designed using the archival SSP to an approximate mode filter designed using the samples of the IW profile taken across the span of the VLA. The input modes for both beampatterns are the exact modes corresponding to the IW SSP. In this case the approximate mode filter has lower crosstalk. Given a choice between using an exact filter computed using an archival profile and an approximate filter computed using measurements of the SSP across the VLA, this example indicates that it is better to use the approximate filter.

V. Summary

This paper presented a technique for designing mode filters using modeshapes from a uniformly-valid WKB-type approximation. Examples for three deep water environments demonstrated that the approximate modes show good agreement with the exact modes. Approximate mode filters have higher crosstalk than exact mode filters designed for the same SSP. The final example in Sec. IV demonstrated that the approximate mode filter may perform better than the exact filter when the approximate filter can be designed using a measurement of the the SSP across the VLA and the exact filter can only be computed for an archival SSP.
Fig. 6. (a) Approximate mode beampatterns for the three deep water environments. Each PI filter is designed for 10 modes using the Miller/Good approximate modeshapes. The input replicas are the exact modeshapes determined from the Prufer code.

Fig. 7. Comparison of mismatch beampatterns for the three deep water environments. Each PI filter is designed for 10 modes using the Miller/Good approximate modeshapes. The input replicas are the exact modeshapes determined from the Prufer code.

Fig. 8. Sound speed perturbations due to internal waves. The blue solid curve is the WOA archival SSP and the red dashed curve is one realization of the archival SSP plus internal waves (IW). A measured SSP (green dash-dot curve) is included for reference.

Fig. 9. Comparison of exact and approximate mode filters. The approximate filter is designed using the true (IW) SSP, and the exact filter is designed using the archival (WOA) SSP shown in Fig. 8. The input modes are the exact modes for the IW profile.

REFERENCES


