2.16. Explain the difference between a fermion and a boson, and give two examples of each.

**Solution:** Particles with integral (in units of \( \hbar \)) spin are bosons. Examples are photons and phonons. Particles with half-integral spin are fermions. Examples are electrons, protons, and neutrons.

## 3 Problems Chapter 3: Quantum Mechanics of Electrons

### 3.1. For the matrix operator

\[
L = \begin{bmatrix}
-5 & 0 \\
1 & 2
\end{bmatrix}
\]

show that eigenvalues and eigenvectors are

\[
\lambda = 2, \quad x = \begin{bmatrix}
0 \\
\alpha
\end{bmatrix},
\]

\[
\lambda = -5, \quad x = \begin{bmatrix}
-7\beta \\
\beta
\end{bmatrix},
\]

where \( \alpha, \beta \neq 0 \). That is, show that the preceding quantities satisfy the eigenvalue problem \( Lx = \lambda x \).

**Solution:**

\[
Lx = \lambda x
\]

\[
\begin{bmatrix}
-5 & 0 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
0 \\
\alpha
\end{bmatrix} = 2 \begin{bmatrix}
0 \\
\alpha
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
2\alpha
\end{bmatrix} = \begin{bmatrix}
0 \\
2\alpha
\end{bmatrix}
\]

\[
Lx = \lambda x
\]

\[
\begin{bmatrix}
-5 & 0 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
-7\beta \\
\beta
\end{bmatrix} = -5 \begin{bmatrix}
-7\beta \\
\beta
\end{bmatrix}
\]

\[
\begin{bmatrix}
35\beta \\
-5\beta
\end{bmatrix} = \begin{bmatrix}
35\beta \\
-5\beta
\end{bmatrix}
\]

### 3.2. Consider the set of functions \( \left\{ \frac{1}{\sqrt{2\pi}} e^{inx}, \ n = 0, \pm 1, \pm 2, \ldots \right\} \).

(a) Show that this is an orthonormal set on the interval \((-\pi, \pi)\).

**Solution:**

\[
\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{inx} \frac{1}{\sqrt{2\pi}} e^{-inx} dx = \frac{1}{2\pi} e^{i(n-m)x} - e^{-i(n-m)x} \frac{i (n - m)}{i (n - m)} = 0 \quad \text{if } n \neq m
\]

\[
\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{inx} \frac{1}{\sqrt{2\pi}} e^{-inx} dx = 1 \quad \text{if } n = m
\]

(b) On the interval \((-\pi/2, \pi/2)\), is the set an orthogonal set, an orthonormal set, or neither?

**Solution:** Evaluating the integral one sees that the set is not orthogonal (and, hence, can’t be orthonormal).

### 3.3. Consider the set of functions \( \left\{ \sqrt{\frac{2}{\pi}} \sin(nx), \ n = 1, 2, \ldots \right\} \) on the interval \((0, \pi)\).

(a) Show that this is an orthonormal set.
Solution:

\[
\int_0^\pi \sqrt{\frac{2}{\pi}} \sin(nx) \sqrt{\frac{2}{\pi}} \sin(mx)dx = \frac{1}{\pi} \int_0^\pi (\cos(mx - nx) - \cos(mx + nx)) \, dx
\]

\[= \frac{1}{\pi} \left( \frac{\sin((n - m)\pi)}{n - m} - \frac{\sin((n + m)\pi)}{n + m} \right) = 0 \quad \text{if} \ n \neq m
\]

\[= \frac{2}{\pi} \int_0^\pi \sin^2(nx)dx = \frac{2}{\pi} \int_0^\pi \left( \frac{1}{2} - \frac{1}{2} \cos 2nx \right) \, dx
\]

\[= \frac{2}{\pi} \left( \frac{1}{2\pi} \right) = 1 \quad \text{if} \ n = m
\]

(b) Determine an operator (including boundary conditions) for which the preceding set are eigenfunctions. What are the eigenvalues?

**Solutions:** The operator is

\[\hat{\mathcal{D}} = \frac{d^2}{dx^2}, \quad (37)
\]

the second derivative operator \((-d^2/dx^2 \text{ also works})\), acting on functions defined over \(0 \leq x \leq \pi\). Every function \(\psi(x) = A \sin(kx) + B \cos(kx)\) is an eigenfunction with eigenvalue \(\lambda = -k^2\), since

\[\frac{d^2}{dx^2} \left( A \sin(kx) + B \cos(kx) \right) = -k^2 \left( A \sin(kx) + B \cos(kx) \right). \quad (38)
\]

If \(k\) is to be an integer, \(k = n\), and \(B = 0\), then the boundary condition is \(\psi(0) = \psi(\pi) = 0\). The eigenvalues are simply \(n = 0, 1, 2, \ldots\).

3.4. For the differential operator \(L = -d^2/dx^2, \ u(0) = u(a) = 0\), determine eigenvalues \(\lambda\) and eigenfunctions \(u\). That is, solve

\[Lu = \lambda u, \quad (39)
\]

where \(u(x)\) is a nonzero function subject to the given boundary conditions. Normalize the eigenfunctions, and show that the eigenfunctions are orthonormal.

**Solution:**

\[-\frac{d^2}{dx^2}u - \lambda u = 0 \quad (40)
\]

\[u = A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x
\]

check: \(-\frac{d^2}{dx^2} \left( A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x \right) - \lambda \left( A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x \right) = 0
\]

\[u(0) = B = 0
\]

\[u(a) = A \sin \sqrt{\lambda}a = 0 \rightarrow \sqrt{\lambda}a = n\pi, \ n = 0, 1, 2, \ldots
\]

\[\lambda = \left( \frac{n\pi}{a} \right)^2.
\]

Therefore

\[u = A \sin \frac{n\pi}{a}x \quad (41)
\]

To normalize the eigenfunctions,

\[\int_0^a A^2 \sin^2 \left( \frac{n\pi}{a}x \right) dx = 1 \quad (42)
\]

\[A^2 \left( \int_0^a \left( \frac{1}{2} - \frac{1}{2} \cos \frac{2\pi}{a}nx \right) \, dx \right) = 1
\]

\[A^2 \left( \frac{1}{2a} \right) = 1 \rightarrow A = \sqrt{\frac{2}{a}}
\]
Finally,
\[
\frac{2}{a} \int_0^a \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{a} x\right) dx = \begin{cases} 
0 & \text{if } n \neq m \\
1 & \text{if } n = m.
\end{cases}
\] (43)

3.5. Repeat problem 3.4, but for boundary conditions \( u'(0) = u'(a) = 0, \) where \( u' = du/dx. \)

**Solution:**
\[
-\frac{d^2}{dx^2} u - \lambda u = 0
\] (44)
\[
u = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x
\]
\[
u'(0) = A = 0
\]
\[
u'(a) = -B \sqrt{\lambda} \sin \sqrt{\lambda} a = 0 \rightarrow \sqrt{\lambda} a = n\pi, \quad n = 0, 1, 2, ...
\]
\[
\lambda = \left(\frac{n\pi}{a}\right)^2.
\]
Therefore
\[
u = B \cos \frac{n\pi}{a} x.
\] (45)

To normalize the eigenfunctions, set
\[
\int_0^a B^2 \cos^2 \left(\frac{n\pi}{a} x\right) dx = 1
\] (46)
leading to
\[
B = \sqrt{\frac{\varepsilon_n}{a}},
\] (47)
where
\[
\varepsilon_n \equiv \begin{cases} 
1, & n = 0 \\
2, & n \neq 0
\end{cases}.
\] (48)

3.6. Assume that some observable of a certain system is measured and found to be \( \lambda_n \) for some integer \( n. \)

By postulate 2, we know that immediately after the measurement the system is in state \( \psi_n, \) which is an eigenstate of the measurement operator \( \hat{\sigma} \) (i.e., \( \hat{\sigma} \psi_n = \lambda_n \psi_n \)).

(a) What can we conclude about the system’s state immediately before the measurement?

**Solution:** Nothing, other than that it was in a superposition with some content in \( \psi_n. \)

(b) Assume that the identical measurement is then performed on 100,000 identical systems, and each time the measurement result is the same, \( \lambda_n. \) What can we infer about the system’s state immediately before the measurement?

**Solution:** We can reasonably assume that before the measurement the system was in state \( \psi_n. \)

3.7. Assume that an electronic state has a lifetime of \( 10^{-8} \) s. What is the minimum uncertainty in the energy of an electron in this state?

**Solution:** From
\[
\Delta E \Delta t \geq \frac{\hbar}{2},
\] (49)
then
\[
\Delta E \geq \frac{\hbar}{2 \Delta t} = \frac{\hbar}{2 (10^{-8})} = \frac{5.273 \times 10^{-27}}{|q_e|} J = 3.291 \times 10^{-8} \text{ eV}.
\] (50)
3.8. In the example of solving the one-dimensional Schrödinger equation on p. 65, we obtained the state functions

$$\Psi(x, t) = \psi(x) e^{-iE_n t / \hbar}$$

where

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin \left(\frac{n\pi}{L} x\right), \quad n \text{ even},$$

$$= \left(\frac{2}{L}\right)^{1/2} \cos \left(\frac{n\pi}{L} x\right), \quad n \text{ odd},$$

are eigenfunctions of the second derivative operator $d^2 / dx^2$, and where energy eigenvalues were found to be

$$E_n = \hbar^2 \left(\frac{n\pi}{L}\right)^2.$$  

(a) Show that the odd eigenfunction (sine) can be written as

$$\psi(x) = \frac{1}{\sqrt{2}} \left[ \frac{1}{i\sqrt{L}} e^{i\frac{n\pi}{L} x} - \frac{1}{i\sqrt{L}} e^{-i\frac{n\pi}{L} x} \right]$$  

and determine a similar expression for the even eigenfunction. The term $\psi_+ (\psi_-)$ represents a wave propagating with positive (negative) momentum. Thus, any state described by sine and cosine can be thought of as representing a superposition of positive and negative momentum states.

**Solution:**

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin \left(\frac{n\pi}{L} x\right) = \left(\frac{2}{L}\right)^{1/2} \left( \frac{e^{i\frac{n\pi}{L} x} - e^{-i\frac{n\pi}{L} x}}{2i} \right),$$

and using

$$k = \frac{n\pi}{L},$$

it was shown in the example that

$$p = \frac{\hbar}{\lambda} = \frac{\hbar k}{2\pi} = \frac{n\pi\hbar}{L} = p_n,$$

so that

$$\psi(x) = \frac{1}{i} \left(\frac{1}{2L}\right)^{1/2} \left( e^{i\frac{n\pi}{L} x} - e^{-i\frac{n\pi}{L} x} \right).$$

(b) Although the decomposition of a standing wave into two counterpropagating waves, as in part (a), is useful, it can be misinterpreted. Since the probability density $\psi(x, t)$ is independent of time, the expectation value of position, $\langle x \rangle$, is independent of time, and so, really, we should not think of the particle as “bouncing” back and forth in the confined space (otherwise, $\langle x \rangle$ would be a function of $t$). Determine the expectation value of momentum, using either (3.217) or (3.219), and discuss your answer in light of the above comment.

**Solution:** The expectation value for momentum is,

$$\langle p_x \rangle = \int_{-L/2}^{L/2} \psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \psi \, dx$$

$$= \int_{-L/2}^{L/2} \left( \frac{1}{\sqrt{2}} \left[ \frac{1}{i\sqrt{L}} e^{-i\frac{n\pi}{L} x} - \frac{1}{i\sqrt{L}} e^{i\frac{n\pi}{L} x} \right] \right)$$

$$\times \left( -i\hbar \frac{\partial}{\partial x} \right) \left( \frac{1}{\sqrt{2}} \left[ \frac{1}{i\sqrt{L}} e^{i\frac{n\pi}{L} x} - \frac{1}{i\sqrt{L}} e^{-i\frac{n\pi}{L} x} \right] \right) \, dx$$

$$= \frac{i}{L} p_n \int_{-L/2}^{L/2} \left( \sin \left( \frac{2\pi n}{L} x \right) \right) \, dx = 0.$$
Therefore, the average momentum is zero, meaning that there are an equal number of positive and negative momentum states, and no net movement.

(c) Assume that the particle is in a state composed of the first two eigenfunctions,

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left( \left( \frac{2}{L} \right)^{1/2} \cos \left( \frac{\pi}{L} x \right) e^{-i \frac{E_1}{\hbar} t} + \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{2\pi}{L} x \right) e^{-i \frac{E_2}{\hbar} t} \right).$$

Show that the expectation value of position as a function of time is

$$\langle x \rangle = \frac{16L}{9\pi^2} \cos \left( \frac{3\hbar \pi^2}{2mL^2 t} \right).$$

Interpret this solution, compared with the expectation value of position for a single stationary state $\psi_n$, which is time-independent.

**Solution:**

$$\langle x \rangle = \int_{-L/2}^{L/2} \psi^* x \psi dx$$

$$= \frac{1}{2L} \int_{-L/2}^{L/2} x \left( \cos \left( \frac{\pi}{L} x \right) e^{i\frac{E_1}{\hbar} t} + \sin \left( \frac{2\pi}{L} x \right) e^{i\frac{E_2}{\hbar} t} \right) \times \left( \cos \left( \frac{\pi}{L} x \right) e^{-i\frac{E_1}{\hbar} t} + \sin \left( \frac{2\pi}{L} x \right) e^{-i\frac{E_2}{\hbar} t} \right) dx$$

$$= \frac{1}{L} \int_{-L/2}^{L/2} x \cos^2 \left( \frac{\pi}{L} x \right) \cos \left( \frac{2\pi}{L} x \right) \sin \left( \frac{2\pi}{L} x \right) e^{i\frac{(E_1 - E_2)}{\hbar} t} dx$$

$$= \frac{2L}{9\pi^2} \cos \left( \frac{3\hbar \pi^2}{2mL^2 t} \right),$$

where we used the fact that the integral of an odd function over symmetric limits is zero. Since

$$\frac{(E_1 - E_2)}{\hbar} = \frac{1}{\hbar} \left( \frac{\hbar^2}{2m} \left( \frac{\pi}{L} \right)^2 - \frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2 \right)$$

$$= \frac{\hbar}{2m} \left( \left( \frac{\pi}{L} \right)^2 - \left( \frac{2\pi}{L} \right)^2 \right) = -\frac{3\hbar \pi^2}{2mL^2},$$

then

$$\langle x \rangle = \frac{16L}{9\pi^2} \cos \left( \frac{3\hbar \pi^2}{2mL^2 t} \right).$$

3.9. Since Schrödinger’s equation is a homogeneous equation, the most general solution for the state function is a sum of homogeneous solutions (3.142),

$$\Psi(r,t) = \sum_n a_n \psi_n(r) e^{-iE_n t/\hbar}.$$

Show that if $\Psi(r,0)$ is known then an expression for the weighting amplitudes $a_n$ can be determined. Assume that the eigenfunction $\psi_n$ form an orthonormal set. Hint: multiply

$$\Psi(r,0) = \sum_n a_n \psi_n(r)$$

(65)
by $\psi_n^*(r)$ and integrate. What is the interpretation of $|a_n|^2$?

**Solution:**

\[
\int \psi_m^* (r) \Psi (r, 0) \, dr^3 = \sum_n a_n \int \psi_m (r) \psi_n^* (r) \, dr^3
\]

\[
= \sum_n a_n \delta_{nm} = a_m.
\]

So, since

\[
|a_n|^2 = \left| \int \psi_n^* (r) \Psi (r, 0) \, dr^3 \right|^2
\]

and

\[
P(\lambda_n) = \left| \int \Psi (r, t) \psi_n^* (x) \, dx \right|^2,
\]

then $|a_n|^2$ is the probability that a measurement will find that the initial state of the particle is $n$.

3.10. Consider a particle with time-independent potential energy, and assume that the initial state of the particle is

\[
\Psi (r, t) = a_1 \psi_1 (r, t) + a_2 \psi_2 (r, t),
\]

such that $P(\lambda_1) = |a_1|^2 = P_1$, $P(\lambda_2) = |a_2|^2 = P_2$, and $|a_1|^2 + |a_2|^2 = 1$. Show that

\[
\langle E \rangle = P_1 \langle E_1 \rangle + P_2 \langle E_2 \rangle.
\]

**Solution:**

\[
\langle E \rangle = \int \Psi^* (r, t) (i\hbar) \frac{\partial}{\partial t} \Psi (r, t) \, d^3r
\]

\[
= \int (a_1^* \psi_1^* (r, t) + a_2^* \psi_2^* (r, t)) (i\hbar) \frac{\partial}{\partial t} (a_1 \psi_1 (r, t) + a_2 \psi_2 (r, t)) \, d^3r
\]

\[
= |a_1|^2 \int \psi_1^* (r, t) (i\hbar) \frac{\partial}{\partial t} \psi_1 (r, t) \, d^3r + |a_2|^2 \int \psi_2^* (r, t) (i\hbar) \frac{\partial}{\partial t} \psi_2 (r, t) \, d^3r
\]

\[
= P_1 \langle E_1 \rangle + P_2 \langle E_2 \rangle.
\]

3.11. For the example of solving the one-dimensional Schrödinger’s equation on p. 65, determine the probability of observing the particle near the boundary wall, $x = \pm L/2$. If the particle is in the $n = 2$ state, where is the particle most likely to be found?

**Solution:** At $x = \pm L/2$, $\psi = 0$, so probability is zero. It would be better to say that the wavefunction is very small near the boundary, and thus, as one considers a region near the boundary, it is very unlikely to find the particle there.

If the particle is in the $n = 2$ state, where is the particle most likely to be found? Even though the expectation value of position is zero,

\[
\langle x \rangle = \int_{-L/2}^{L/2} \psi^* (x, t) x \psi (x, t) \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} x \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = 0,
\]

the $n = 2$ state peaks away from the origin. The particle is most likely to be found where the wavefunction is largest,

\[
\frac{d}{dx} \sin \left( \frac{n\pi x}{L} \right) = \frac{\pi}{L} n \cos \frac{\pi}{L} nx = 0
\]

\[
\rightarrow \cos \frac{\pi}{L} 2x = 0
\]

\[
\rightarrow x = \pm \frac{L}{4}
\]
3.12. For the example of solving the one-dimensional Schrödinger’s equation on p. 65, assume that the particle is in the \( n = 2 \) state. What is the probability that a measurement of energy will yield

\[
E_2 = \frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2
\]

\( \text{(75)} \)

**Solution:** Since the particle is already in the \( n = 2 \) state, by postulate 2 the probability that an energy measurement will yield \( E_2 \) is 100%.

What is the probability that a measurement of energy will yield

\[
E_3 = \frac{\hbar^2}{2m} \left( \frac{3\pi}{L} \right)^2
\]

\( \text{(76)} \)

**Solution:** Since the particle is already in the \( n = 2 \) state, by postulate 2 the probability that an energy measurement will yield \( E_3 \) is 0%. This can easily be seen from orthogonality,

\[ P(E_3) = \left( \frac{2}{L} \right)^2 \left| \int_{-L/2}^{L/2} \sin \left( \frac{2\pi}{L}x \right) \cos \left( \frac{3\pi}{L}x \right) dx \right|^2 = 0. \]

\( \text{(77)} \)

3.13. Consider a quantum encryption scheme using photons. Assume that a photon can only exist in either state 1, \( \psi_1 \), having energy \( E_1 \), or state 2, \( \psi_2 \), having energy \( E_2 \), or in a superposition of the two states, \( \Psi = a\psi_1 + b\psi_2 \). Assume that the states are orthonormal.

(a) If a photon exists in the superposition state \( \Psi = a\psi_1 + b\psi_2 \), what is the relationship between \( a \) and \( b \)?

(b) If a photon exists in the superposition state \( \Psi = a\psi_1 + b\psi_2 \), determine the probability of measuring energy \( E_2 \). Show all work, and/or explain your answer.

(c) If the photon in a superposition state is sent over a network, explain how undetected eavesdropping would be impossible.

**Solution:**

(a) Since the sum of probabilities must be unity, then \( |a|^2 + |b|^2 = 1 \).

(b)

\[
P(E_2) = \left| \int_0^a \Psi(x) \psi_2^*(x) dx \right|^2 = \left| \int_0^a \{a\psi_1 + b\psi_2\} \psi_2^*(x) dx \right|^2 = |b|^2.
\]

\( \text{(78)} \)

(c) Undetected eavesdropping is impossible, since any measurement would collapse the state function.

3.14. In Chapter 6, the reflection and transmission of a particle across a potential barrier will be considered. For now, assume that a potential energy discontinuity is present at \( x = a \), and that to the left of the discontinuity the wavefunction is given by

\[
\Psi(x,t) = (e^{ikx} + R e^{-ikx}) e^{-iEt/\hbar},
\]

\( \text{(79)} \)

and to the right of the discontinuity,

\[
\Psi(x,t) = Te^{iqx} e^{-iEt/\hbar},
\]

\( \text{(80)} \)

where \( R \) and \( T \) are reflection and transmission coefficients, respectively, which will depend on the properties of the different regions and on the discontinuity in potential at \( x = a \). Determine the probability current density on either side of the discontinuity.
Solution:

\[ J(r,t) = -\frac{i\hbar}{2m} \left( \Psi^*(r,t) \nabla \Psi (r,t) - \Psi(r,t) \nabla \Psi^*(r,t) \right) \tag{81} \]

\[ = \frac{-i\hbar}{2m} \hat{x} \left( \Psi^*(r,t) \frac{\partial}{\partial x} \Psi (r,t) - \Psi(r,t) \frac{\partial}{\partial x} \Psi^*(r,t) \right) \]

\[ = \frac{-i\hbar}{2m} \hat{x} \left( (e^{-ikx} + R^* e^{ikx}) \frac{\partial}{\partial x} (e^{ikx} + R e^{-ikx}) - (e^{ikx} + R e^{-ikx}) \frac{\partial}{\partial x} (e^{-ikx} + R^* e^{ikx}) \right) \]

\[ = \hat{x} \frac{\hbar k}{m} \left( 1 - |R|^2 \right), \]

on the left, and, on the right,

\[ J(r,t) = -\frac{i\hbar}{2m} \left( \Psi^*(r,t) \nabla \Psi (r,t) - \Psi(r,t) \nabla \Psi^*(r,t) \right) \tag{82} \]

\[ = -\frac{i\hbar}{2m} \hat{x} \left( \Psi^*(r,t) \frac{\partial}{\partial x} \Psi (r,t) - \Psi(r,t) \frac{\partial}{\partial x} \Psi^*(r,t) \right) \]

\[ = -\frac{i\hbar}{2m} \hat{x} \left( (T^* e^{-iqx}) \frac{\partial}{\partial x} (T e^{iqx}) - (T e^{iqx}) \frac{\partial}{\partial x} (T^* e^{-iqx}) \right) \]

\[ = \hat{x} \frac{\hbar q}{m} |T|^2. \]

3.15. In the example of solving the one-dimensional Schrödinger’s equation on p. 65, we obtained the state functions

\[ \Psi(x,t) = \psi(x) e^{-iE_n t/\hbar} \tag{83} \]

where

\[ \psi(x) = \begin{cases} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{n\pi x}{L} \right), & n \text{ even,} \\ \left( \frac{2}{L} \right)^{1/2} \cos \left( \frac{n\pi x}{L} \right), & n \text{ odd,} \end{cases} \tag{84} \]

and where

\[ E_n = \frac{\hbar^2}{2m} \left( \frac{n\pi}{L} \right)^2. \tag{85} \]

Determine the probability current density. Discuss your result.

**Solution:** Since \( \psi \) is real-valued, the probability current density is zero (since in this case \( \psi = \psi^* \)). This means that there is no net current; these states are called stationary states.

3.16. Assume that the wave function

\[ \psi(z,t) = 200e^{\hat{M}(kz-\omega t)} \tag{86} \]

describes a beam of 2 eV electrons having only kinetic energy. Determine numerical values for \( k \) and \( \omega \), and find the associated current density in A/m.
Solution:

\[ E = \hbar \omega = 2 |q_e| \]  
\[ v = \sqrt{\frac{2E}{m_e}} = \sqrt{\frac{2(2|q_e|)}{m_e}} = 8.387 \times 10^5 \text{ m/s} \]  
\[ \lambda = \frac{\hbar}{p} = \frac{\hbar}{m_e v} = \frac{\hbar}{m_e (8.387 \times 10^5)} = 0.8673 \text{ nm} \]  
\[ k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8673 \times 10^{-5}} = 7.244 \times 10^9 \text{ m}^{-1} \]

so, \( \psi(z,t) = 200e^{ikz}e^{-i\omega t} = 200e^{i(7.244 \times 10^9)z}e^{-i(\frac{2\omega}{\lambda})t}, \)

and

\[ J(z,t) = \frac{-i\hbar}{2m_e} \left( \psi^*(z,t) \frac{\partial}{\partial z} \psi(z,t) - \psi(z,t) \frac{\partial}{\partial z} \psi^*(z,t) \right) \]  
\[ = \frac{-i\hbar}{2m_e} \left( 200e^{-i(7.244 \times 10^9)z}e^{i(\frac{2\omega}{\lambda})t} \frac{\partial}{\partial z} (200e^{i(7.244 \times 10^9)z}e^{-i(\frac{2\omega}{\lambda})t}) \right. \]  
\[ - \left. 200e^{i(7.244 \times 10^9)z}e^{-i(\frac{2\omega}{\lambda})t} \frac{\partial}{\partial z} (200e^{-i(7.244 \times 10^9)z}e^{i(\frac{2\omega}{\lambda})t}) \right) = \hbar 3.355 \times 10^{10} \text{ A/m}. \]

4 Problems Chapter 4: Free and Confined Electrons

4.1. Write down the wavefunction \( \psi(z,t) \) for a 3 eV electron in an infinite space, travelling along the positive \( z \) axis. Assume that the electron has only kinetic energy. Plug your answer into Schrödinger’s time-dependent equation to verify that it is a solution.

Solution:

\[ E = \hbar \omega = 3 |q_e| \]  
\[ v = \sqrt{\frac{2E}{m_e}} = \sqrt{\frac{2(3|q_e|)}{m_e}} = 1.027 \times 10^6 \text{ m/s} \]  
\[ \lambda = \frac{\hbar}{p} = \frac{\hbar}{m_e v} = \frac{\hbar}{m_e (1.027 \times 10^6)} = 0.7082 \text{ nm} \]  
\[ k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.7082 \times 10^{-5}} = 8.872 \times 10^9 \text{ m}^{-1} \]

or, use (4.5), \( k = \sqrt{\frac{2m_e}{\hbar^2}} 3|q_e| = 8.87 \times 10^9 \)

so, \( \psi(z,t) = Ae^{ikz}e^{-i\omega t} = Ae^{i(8.872 \times 10^9)z}e^{-i(\frac{3\omega}{4})t} \)

\[ i\hbar \frac{\partial \psi(z,t)}{\partial t} = \left( -\frac{\hbar^2}{2m_e} \frac{d^2}{dz^2} + V \right) \psi(z,t). \]

4.2. Determine the wavefunction \( \psi(z,t) \) for a 3 eV electron in an infinite space, travelling along the \( z \) axis at a velocity of \( 10^5 \text{ m/s} \). Determine the particle’s potential energy, and plug your answer into Schrödinger’s time-dependent equation to verify that it is a solution.